A numerical method for a depth averaged Euler system in two dimensions

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Motivations

Better understand dispersive phenomena in order to better predict geophysical situations.

(a) Dam break

(b) Tsunami

(c) Mascaret

(d) Hydraulic jump
Dispersive effects: Dingemans experiment

Hydrostatic model

Non-hydrostatic model
A depth-averaged model

\[
\frac{\partial H}{\partial t} + \frac{\partial H u}{\partial x} + \frac{\partial H v}{\partial y} = 0
\]

\[
\frac{\partial H u}{\partial t} + \frac{\partial}{\partial x}(H u^2) + \frac{\partial}{\partial y}(H u v) + \frac{\partial}{\partial x}(g \frac{H^2}{2}) + \frac{\partial}{\partial y}(H p_{nh}) = -(gH + 2p_{nh}) \frac{\partial z_b}{\partial x}
\]

\[
\frac{\partial H v}{\partial t} + \frac{\partial}{\partial y}(H v^2) + \frac{\partial}{\partial x}(H u v) + \frac{\partial}{\partial y}(g \frac{H^2}{2}) + \frac{\partial}{\partial x}(H p_{nh}) = -(gH + 2p_{nh}) \frac{\partial z_b}{\partial y}
\]

\[
\frac{\partial H w}{\partial t} + \frac{\partial H w u}{\partial x} + \frac{\partial H w v}{\partial y} = 2p_{nh}
\]

\[
-\frac{\partial H u}{\partial x} - \frac{\partial H v}{\partial y} + u \frac{\partial (H + 2z_b)}{\partial x} + v \frac{\partial (H + 2z_b)}{\partial y} = 2\bar{w}
\]

- Average velocity
  \(u(x, t) = \frac{1}{H} \int_{z_b}^{\eta} u(x, z, t) \, dz\),

- Green-Naghdi class model

- Energy balance
  \(\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(u(E + g \frac{H^2}{2} + H p_{nh})) = 0\)
Objective: Apply a projection-correction scheme

- Rewrite the model as Euler
- Define the operators \( \text{div}_{sw} \) and \( \nabla_{sw} \):

\[
\nabla_{sw} (f) = \left( \begin{array}{c}
H \frac{\partial f}{\partial x} + f \frac{\partial (H+2z_b)}{\partial x} \\
H \frac{\partial f}{\partial y} + f \frac{\partial (H+2z_b)}{\partial y} \\
-2f
\end{array} \right)
\]

\[
\text{div}_{sw} (\mathbf{v}) = \frac{\partial H v_1}{\partial x} + \frac{\partial H v_2}{\partial y} - v_1 \frac{\partial (H + 2z_b)}{\partial x} - v_2 \frac{\partial (H + 2z_b)}{\partial y} + 2v_3
\]

which verify:

\[
\int_{\Omega} \text{div}_{sw} (\mathbf{v}) f \, dx = -\int_{\Omega} \nabla_{sw} (f) \cdot \mathbf{v} \, dx + \int_{\partial \Omega} H f \mathbf{v} \cdot \mathbf{n} \, d\sigma
\]

- Remark that the operators are \( H \) and \( z_b \) dependent.
Euler formulation

We can rewrite the model:

\[
\frac{\partial H}{\partial t} + \nabla_0 \cdot (H \mathbf{u}) = 0
\]

\[
\frac{\partial H \mathbf{u}}{\partial t} + \nabla_0 \cdot (H \mathbf{u} \otimes \mathbf{u}) + \nabla_0 \left( \frac{g}{2} H^2 \right) + \nabla_{sw} (p) = -g H \nabla_0 (z_b)
\]

\[
\text{div}_{sw} (\mathbf{u}) = 0
\]

where we note \( \mathbf{u} = \begin{pmatrix} u \\ w \end{pmatrix} \) the velocity of the Depth-averaged Euler model and

\[
\nabla_0 = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ 0 \end{pmatrix}
\]
Fractional time scheme with incremental version

- **Hyperbolic step**

\[
H_{i}^{n+1/2} = H_{i}^{n} - \sum_{j \in K_{i}} \sigma_{ij} F_{H}(H_{i}^{n}, H_{j}^{n}) - \sigma_{i} F_{H}(H_{i}^{n}, H_{e,i}^{n})
\]

\[
(Hu)^{n+1/2}_{i} = (Hu)^{n}_{i} - \sum_{j \in K_{i}} \sigma_{ij} F_{(Hu)}((Hu)^{n}_{i}, (Hu)^{n}_{j}) - \sigma_{i} F_{(Hu)}((Hu)^{n}_{i}, (Hu)^{n}_{e,i})
\]

- **Correction step**

\[
H_{i}^{n+1} = H_{i}^{n+1/2}
\]

\[
(Hu)^{n+1}_{i} = (Hu)^{n+1/2}_{i} - \Delta t \nabla_{sw}(p^{n+1})_{i}
\]

\[
\text{div}_{sw}(u^{n+1}) = 0.
\]
Correction step: Variational formulation

- Correction step: solve

\[
\begin{align*}
H^{n+1} &= H^{n+1/2} \\
(Hu)^{n+1} + \Delta t \nabla_{sw} p^{n+1} &= (Hu)^{n+1/2} \\
\text{div}_{sw} (u^{n+1}) &= 0
\end{align*}
\]

### Variational mixed problem

Find \( u \in V \) and \( p \in Q \) such that

\[
\frac{1}{\Delta t^n} a(u, v) + b(v, p) = \frac{1}{\Delta t^n} a(u^{n+1/2}, v), \quad \forall v \in V,
\]

\[
b(u, q) = 0, \quad \forall q \in Q.
\]

with

\[
Q = \{ q \in L^2(\Omega) \},
\]

and

\[
V = \{ v = (v_1, v_2) \in (L^2(\Omega))^3 | \text{div}_{sw} (v) \in L^2(\Omega) \},
\]

\[
a(u, v) = \int_{\Omega} Hu \cdot v \, dx, \quad \forall u, v \in V,
\]

\[
b(v, q) = -\int_{\Omega} \text{div}_{sw} (v) q \, dx, \quad \forall v \in V, \forall q \in Q,
\]
Correction step: Variational formulation

- Correction step: solve

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\begin{align*}
H^{n+1} &= H^{n+1/2} \\
(Hu)^{n+1} + \Delta t \nabla_{sw} p^{n+1} &= (Hu)^{n+1/2} \\
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\]

- Pressure equation

Shallow water version of the Laplacian equation

\[
\Delta_{sw} p = \frac{1}{\Delta t^n} \text{div}_{sw} (u^{n+1/2})
\]

with

\[
\Delta_{sw} p = \text{div}_{sw} \left( \frac{1}{H} \nabla_{sw} p \right).
\]
Correction step: Variational formulation

- Correction step: solve

\[
\begin{align*}
H^{n+1} &= H^{n+1/2} \\
(Hu)^{n+1} + \Delta t \nabla_{sw} p^{n+1} &= (Hu)^{n+1/2} \\
\text{div}_{sw} (u^{n+1}) &= 0
\end{align*}
\]

**Variational mixed problem**

Find \( u \in V \) and \( p \in Q \) such that

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\frac{1}{\Delta t^n} a(u, v) + b(v, p) = \frac{1}{\Delta t^n} a(u^{n+1/2}, v), \quad \forall v \in V,
\]

\[
b(u, q) = 0, \quad \forall q \in Q.
\]

- Compatibility of the boundary conditions

**Hyperbolic part**

- Distinguish Fluvial flow / Torrential flow
- Consider a Riemann problem at the interface

**Correction part**

- Dirichlet or Neumann boundary condition for the pressure
- Slide boundary condition for the velocity
Numerical approximation

Discrete problem

\[
\begin{pmatrix}
\frac{1}{\Delta t^n} A_H & B^t \\
B & 0
\end{pmatrix}
\begin{pmatrix}
U \\
P
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{\Delta t^n} A_H U^{n+1/2} \\
0
\end{pmatrix},
\]

\[
M_H = \left( \int_{\Omega} H \varphi_i \varphi_j dx \right)_{1 \leq i, j \leq N}, \quad A_H = \begin{pmatrix}
M_H & 0 & 0 \\
0 & M_H & 0 \\
0 & 0 & M_H
\end{pmatrix}.
\]

and the two matrices \( B^t, B \) defined by

\[
B^t = \left( \int_{\Omega} \nabla_{sw}(\phi_l) \varphi_i dx \right)_{1 \leq l \leq M, 1 \leq i \leq N}, \quad B = - \left( \int_{\Omega} \text{div}_{sw}(\varphi_j) \phi_l dx \right)_{1 \leq l \leq M, 1 \leq j \leq N}.
\]
2D finite volume/ Finite element method

- Choice of a stable pair of finite elements.

![Dual mesh diagram](image)

**Figure**: Representation of the dual mesh (finite volume mesh)

- Iterative method to solve the linear problem: Conjugate Gradient - Uzawa algorithm: $U^{(0)}, P^{(0)}$ given, while $\|BU^{(k)}\| > \text{tolerance}$:

\[
\text{solve } A_H U^{(k+1)} = A_H U^{n+1/2} - \Delta t B^t P^{(k)} \\
\text{compute } P^{(k+1)} = P^{(k)} + \alpha BU^{(k)}, \quad k > 0
\]
Properties of the scheme from the 1D scheme

- **Positivity of \( H \)**

  CFL condition of the kinetic scheme for the shallow water model \( \Rightarrow \) Positivity of \( H^{n+1/2} \)

  Splitting scheme \( \Rightarrow H^{n+1} = H^{n+1/2} \)

- **The lake at rest**

  Lake at rest \( \Leftrightarrow Hu = 0 \) and \( H + z_b = Cste \)

  Elliptic equation \( \Rightarrow p = 0 \)

  Satisfied by the hydrostatic part with the hydrostatic reconstruction.

- **Discrete entropy in time**

- **Inf-sup condition**

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Numerical test for an analytical solution

- Propagation of a solitary wave

\[ H = H_0 + a \left( \text{sech} \left( \frac{x - ct}{l} \right) \right)^2 \]

\[ H_0 = 1.0 \text{ m}, \ a = 0.2 \text{ m}, \ c, l \in \mathbb{R} \]
**Convergence rate**

- Propagation of a solitary wave

\[ H = H_0 + a \left( \text{sech} \left( \frac{x - ct}{l} \right) \right)^2 \]

\[ H_0 = 1.0 \, \text{m} , \, a = 0.3 \, \text{m} , \, c, l \in \mathbb{R} \]
Wet/dry interface

Pressure equation

\[ \Delta_{sw} p = \frac{1}{\Delta t^n} \text{div}_{sw} (u^{n+1/2}) \quad \text{with} \quad \Delta_{sw} p = \text{div}_{sw} \left( \frac{1}{\max(H, \epsilon)} \nabla_{sw} p \right). \]
Thacker’s solution

- Add a source term for the momentum equation:

\[
\frac{\partial Hw}{\partial t} + \frac{\partial Huw}{\partial x} + \frac{\partial Hvw}{\partial y} - 2p = Hs.
\]

- Solution:

\[
\begin{align*}
H(x, y, t) &= \max(0, H_0 - \frac{\alpha}{2} (x - a \cos(\sqrt{rt}))^2 - \frac{\beta}{2} (y - a \sin(\sqrt{rt}))^2, \\
u(x, y, t) &= -a\sqrt{r} \sin(\sqrt{rt}), \\
v(x, y, t) &= a\sqrt{r} \cos(\sqrt{rt}), \\
w(x, y, t) &= -\alpha a\sqrt{r} \sin(\sqrt{rt})x + \alpha a\sqrt{r} \cos(\sqrt{rt})y, \\
p(x, y, t) &= \frac{a^2 \alpha r}{2} H, \\
s(x, y, t) &= \alpha ar \sin(\sqrt{rt})x - \alpha ar \cos(\sqrt{rt})y, \\
z_b(x, y) &= \frac{\alpha}{2} (x^2 + y^2),
\end{align*}
\]

where \(a, \alpha > 0\) and \(a\alpha < 1\), \(r = \frac{\alpha g}{1-\alpha^2a^2} \).
Thacker’s result

Thacker’s test - Comparison with analytical solution
Convergence rate

- First and second order

(c) Velocity

(d) Water depth

(e) Pressure
## Conclusion and outlook

### Properties
- Discrete entropy
- Equilibria

### Validation of the scheme
- Analytical solution
- Comparison with experimental data 1D

### Two dimensional
- Variational formulation
- Finite element method
- Iterative method

### Outlook
- Higher order
- Geophysical case: Tsunami...
- Optimization
Thank you!