Stability theory for difference approximations of some dispersive shallow water equations

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Introduction: laminar roll waves in laboratory

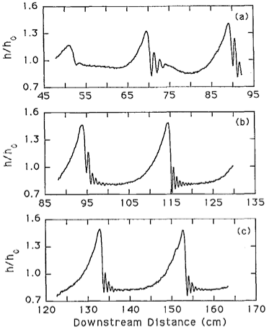
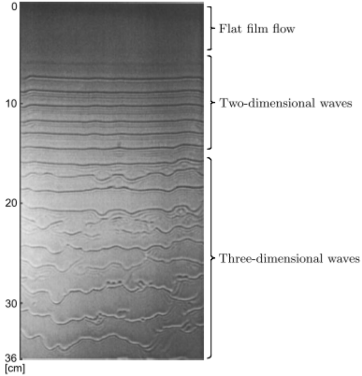


FIG. 3. The evolution of solitary waves forced at $f=1.5$ Hz with $\beta=6.4°$, $R=29$, and Weber number $W=35$. Three wave profiles are measured at increasing distances from the inlet to show the spatial evolution. Phase-sensitive averaging is employed here to reduce imaging noise.



Liu and Gollub experience (Phys of Fluids 94)

Photo of 2-d roll-waves (Park et Nosoko AIChE, 2003)

Outline of the talk

1. Modeling of thin film flows
   - Shallow water equations with surface tension
   - Related models: phase transition
2. Stability of difference approximations for shallow water eqs
   - Von Neumann (linearized) stability
   - Entropy stability (Schrödinger type formulation)
3. Numerical simulations
   - Entropy stability: numerical comparison
   - Liu Gollub experiment (comparison with experimental data)

- Collaboration with J.-P. Vila (IMT Toulouse)

Thin film flows: shallow water equations

- Shallow water flows: the aspect ratio is small $\varepsilon = H/L \ll 1$ (Liu Gollub experiment: $H \sim 1mm, L \sim 1cm$)
- Small Reynolds numbers: $Re \in (1; 100)$
- Surface tension is not negligible (order $O(\varepsilon)$ in non dimensional variables)

First order consistent/conservative model (P.N., J.-P. Vila, 2006)

$$\partial_t h + \partial_x q = 0,$$
$$\partial_t q + \partial_x\left(\frac{q^2}{h} + P(h, A_1)\right) = A_1\left(g\sin(\theta)h - \frac{3\nu q}{h^2} + \frac{\sigma}{\rho}h\partial_{xxx}h\right) + 4\nu\partial_{xx}q,$$
$$P(h, A_1) = \left(\frac{4}{45} - \frac{2A_1}{25}\right)\left(\frac{g\sin(\theta)}{\nu}\right)^2 h^5 + A_1 g\cos(\theta)\frac{h^2}{2}.$$

Remark: the viscous term is heuristic

Related models: Euler Korteweg equations

Euler-Korteweg equations in conservative variables

$$\partial_t \rho + \partial_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \partial_x\left(\rho u^2 + P(\rho)\right) = \partial_x\left(\rho\kappa(\rho)\partial_{xx}\rho + (\rho\kappa'(\rho) - \kappa(\rho))\frac{(\partial_x\rho)^2}{2}\right),$$

- $\kappa(\rho) = constant/\rho$: quantum hydrodynamic (=NLS)
- $\kappa(\rho) = constant$, $P(\rho) = \dfrac{\gamma\rho}{1-\rho} - \rho^2$: Van der Waals gas (phase transition)

Additional Energy equation

$$\partial_t\left(\rho\frac{u^2}{2} + F(\rho) + \kappa(\rho)\frac{(\partial_x\rho)^2}{2}\right) + \partial_x\mathcal{F}(\rho, u, \partial_x\rho, \partial_x u) = 0$$

C. Chalons, P.G. LeFloch *High-Order Entropy-Conservative Schemes and Kinetic Relations for van der Waals Fluids*, JCP (2001): E-K in Lagrangian coordinates of mass/Semi-discrete schemes

Stability of difference schemes: von Neumann stability

- **Remark**: due to the presence of the third order derivative, the energy equation is hardly satisfied in the original formulation
- **A simplified problem**: we check stability for linearized shallow water equations (=Fourier analysis)
- **Interest:** provides necessary and, in practice, sufficient condition of stability

Linearized equations (conservative variables: $v = (h, q)^T$)

$$\partial_t v + A\partial_x v = B\partial_{xxx} v, \quad A = \begin{pmatrix} 0 & 1 \\ \bar{c}^2 - \bar{u}^2 & 2\bar{u} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \bar{\sigma} & 0 \end{pmatrix}.$$

- **Dispersion relation:** $s(k) = \bar{u} \pm \sqrt{\bar{c}^2 + \bar{\sigma} k^2}$
- **Heuristic CFL condition** $s(k)\frac{\delta t}{\delta x} \leqslant 1$. Here $s(k) \sim K/\delta x$ then CFL condition: $\delta t = O(\delta x^2)$.

Von Neumann stability I: formulation of the problem

Stability of difference approximation in the form

$$v_i^{n+1} - v_i^n + \lambda_1 \left(f_{i+\frac{1}{2}}^{n+\theta} - f_{i-\frac{1}{2}}^{n+\theta} \right) = \lambda_3 B \left(v_{i+2}^{n+\theta} - 2v_{i+1}^{n+\theta} + 2v_{i-1}^{n+\theta} - v_{i-2}^{n+\theta} \right). \quad (1)$$

with $\lambda_k = \delta t / \delta x^k$, and $v_i^{n+\theta} = (1-\theta) v_i^n + \theta v_i^{n+1}$.

- Lax-Friedrichs scheme: $f_{i+\frac{1}{2}}^n = \dfrac{Av_i^n + Av_{i+1}^n}{2} - \dfrac{1}{2\lambda_1}(v_{i+1}^n - v_i^n)$
- Rusanov scheme: $f_{i+\frac{1}{2}}^n = \dfrac{Av_i^n + Av_{i+1}^n}{2} - \dfrac{\rho(A)}{2}(v_{i+1}^n - v_i^n)$
- Roe scheme: $f_{i+\frac{1}{2}}^n = \dfrac{Av_i^n + Av_{i+1}^n}{2} - \dfrac{|A|}{2}(v_{i+1}^n - v_i^n)$

Von Neumann stability II: first order accurate schemes

Definition

We search for solutions of (1) in the form $v_i^n = \xi^n e^{-in\theta}$: a scheme is stable in the sense of Von Neumann if $|\xi| \leqslant 1$ for all $\theta \in [0, 2\pi]$

- **Instability of Roe scheme:** The scheme (1) with Roe type flux and $\theta = 0$ (forward Euler time discretization: FE), $\theta = 1$ (backward Euler time discretization: BE) is always unstable: there exists $\eta > 0$ such that if $\lambda_3 < \eta$ and $dx < \eta$, then there exists θ and $\xi_\pm(\theta)$ so that $|\xi_-(\theta)| < 1 < |\xi_+(\theta)|$.

- **Stability of Lax-Friedrichs scheme:**
  - FE time discretization ($\theta = 0$): stable under cfl condition $\delta t = O(\delta x^2)$
  - BE time discretization ($\theta \geqslant 1/2$): inconditionally stable

- **Stability of Rusanov scheme:**
  - FE time discretization ($\theta = 0$): stable under cfl condition $\delta t = O(\delta x^3)$
  - BE time discretization ($\theta \geqslant 1/2$): inconditionally stable

Von Neumann stability III: second order accurate schemes

We use a MUSCL type scheme for space discretization:

$$\begin{aligned}\frac{dv_j}{dt} &- \frac{A}{8\delta x}(v_{j+2} - 6v_{j+1} + 6v_{j-1} - v_{j-2}) \\ &+ \frac{\nu_n}{8\delta x^2}\left(v_{j+2} - 4v_{j+1} + 6v_j - 4v_{j-1} + v_{j-2}\right) \\ &= \frac{B}{\delta x^3}\left(v_{j+2} - 2v_{j+1} + 2v_{j-1} - v_{j-2}\right).\end{aligned}$$

Remark: ν_n is the numerical viscosity (L-F: $\nu_n = \delta x^2/2\delta t$, Ru: $\nu_n = \rho(A)\delta x$)

- **Stability of Lax-Friedrichs scheme:**
  - Runge Kutta 2 : stable under CFL condition $\delta t = O(\delta x^2)$
  - Crank Nicolson ($\theta = 1/2$): inconditionally stable

- **Stability of Rusanov scheme:**
  - Runge Kutta 2 ($\theta = 0$): stable under CFL condition $\delta t = O(\delta x^{7/3})$
  - Crank Nicolson ($\theta = 1/2$): inconditionally stable

Entropy stability of difference schemes: new formulation 1

"Entropy" of the Euler-Korteweg system

$$U(\rho, u, \partial_x \rho) = \int \rho \frac{u^2}{2} + F(\rho) + \kappa(\rho)\frac{(\partial_x \rho)^2}{2}$$

- Not an usual entropy (presence of $\partial_x \rho$): reduction of order needed (see C.W. Shu for KdV type equations with DG methods)
- A natural new variable: $w = \sqrt{\frac{\kappa(\rho)}{\rho}}\partial_x \rho$
- The "entropy" now U reads

$$U(\rho, u, w) = \int \rho \frac{u^2 + w^2}{2} + F(\rho).$$

Entropy stability of difference schemes: new formulation 2

Euler-Korteweg equations: "Schrodinger type formulation"

$$\partial_t v + \partial_x f(v) = \partial_x(B(\rho)\partial_x(\rho^{-1}v)), \quad B(\rho) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu(\rho) \\ 0 & -\mu(\rho) & 0 \end{pmatrix} \quad (2)$$

with $v = (\rho, \rho u, \rho w)^T$, $f(v) = (\rho u, \rho u^2 + P(\rho), \rho u w)^T$.

- The Schrodinger formulation is obtained by setting $\psi = \rho u + i\rho w$ (useful for well posedness: see Benzoni-Danchin-Descombes 2006)
- Setting $U(v) = \rho\frac{u^2+w^2}{2} + F(v)$ and $G(v) = u(U(v) + P(\rho))$:

Energy equation in the new formulation (classic energy estimate)

$$\partial_t U(v) + \partial_x G(v) = \partial_x \left(\mu(\rho)(u\partial_x w - w\partial_x u)\right). \quad (3)$$

Entropy stability of difference scheme: definition

We consider the following semi discretized system (setting $z = \rho^{-1} v$)

$$\frac{d}{dt} v_j(t) + \frac{f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}}{\delta x} = \frac{B(\rho_{j+\frac{1}{2}})\,(z_{j+1} - z_j) - B(\rho_{j-\frac{1}{2}})\,(z_j - z_{j-1})}{\delta x^2}. \qquad (4)$$

Definition

The semi-discretized scheme (4) is entropy stable if there exists a numerical flux $\mathcal{G}_{j+\frac{1}{2}}$, consistent with the entropy flux in (3), so that

$$\frac{d}{dt} U(v_j(t)) + \frac{\mathcal{G}_{j+\frac{1}{2}} - \mathcal{G}_{j-\frac{1}{2}}}{\delta x} \leqslant 0.$$

E. Tadmor *Entropy stability theory for difference approximations of nonlinear conservation laws and related time-dependent problems* Acta Numerica (2003)
P.G. LeFloch, J.M. Mercier, C. Rohde *Fully discrete, entropy conservative schemes of arbitrary order*, SIAM J. Numer. Anal. 40 (2002)

Entropy stability: fully discrete scheme

- By using the convexity of the entropy, one has

Theorem

*Consider the entropy (spatially) stable semi scheme which is a difference approximation of (2) with $B = 0$, then the scheme (**??**) is (unconditionally) entropy stable. There exists $\mathcal{G}^n_{j+\frac{1}{2}}$ so that*

$$U(v_j^{n+1}) - U(v_j^n) + \mathcal{G}^n_{j+\frac{1}{2}} - \mathcal{G}^n_{j-\frac{1}{2}} \leqslant 0, \forall j, \quad \forall n. \tag{5}$$

- For explicit schemes, one has

Theorem

- *Explicit scheme with Lax-Friedrichs flux is entropy stable with CFL $\delta t \ll \delta x^2$*
- *Explicit scheme with Rusanov flux is entropy stable with CFL $\delta t \ll \delta x^3$*

Entropy conservation: a Hamiltonian formulation

Hamiltonian formulation of Euler-Korteweg equations:

$$\mathcal{H}(\rho, u) = \int \rho \frac{u^2}{2} + F(\rho) + \kappa(\rho) \frac{(\partial_x \rho)^2}{2} dx$$

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} = \mathcal{J} \nabla \mathcal{H}(\rho, u), \quad \mathcal{J} = \partial_x J, \quad J = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

- Other examples of Hamiltonian PDE's: NLS, generalized Korteweg de Vries equation, Kawahara equation, more generally water wave models
- A spatial discretization which respects this structure is trivially **entropy conservative**

An Entropy conservative scheme I

- Restriction to periodic boundary conditions: $(\varrho, \mathrm{u}) = (\rho_j, u_j)_{i=1,\cdots,N}$, $(\rho_{j+N}, u_{j+N}) = (\rho_j, u_j)$
- Define for the discretized Hamiltonian

$$\mathrm{H}(\varrho, \mathrm{u}) = \sum_{j=1}^{N} \rho_j \frac{u_j^2}{2} + F(\rho_j) + \frac{1}{2}\kappa(\rho_j)\left(\frac{\rho_{j+1} - \rho_j}{\delta x}\right)^2.$$

- Define a discretized version of $\mathcal{J}$:

$$J = \begin{pmatrix} 0 & -\mathrm{I}_N \\ -\mathrm{I}_N & 0 \end{pmatrix}, \quad Du_j = \frac{u_{j+1} - u_{j-1}}{2\delta x}.$$

- Discretized Hamiltonian system:

$$\frac{d}{dt}\begin{pmatrix} \varrho \\ \mathrm{u} \end{pmatrix} = J \begin{pmatrix} D\nabla_\varrho \mathrm{H}(\varrho, \mathrm{u}) \\ D\nabla_\mathrm{u} \mathrm{H}(\varrho, \mathrm{u}) \end{pmatrix}.$$

An entropy conservative scheme II

- Spatial discretization with **centered difference**: no numerical viscosity!
- **Drawback:** possible numerical instability (Euler time discretization), possible occurrence of spurious oscillatory modes (Crank Nicolson time discretization). Example: discretization of Burgers equation
- BUT: presence of capillarity (control on the gradient of ρ)

Theorem

A fully discrete scheme with backward Euler time discretization is entropy stable.

- **Remark**: any explicit method is unstable
- **Question**: Time discretization preserving the discrete Hamiltonian?

Theorem

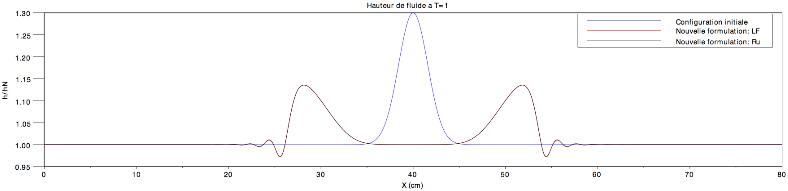
The Crank Nicolson time discretization preserves the Hamiltonian for linearized Euler-Korteweg equations

Entropy stability: numerical comparison I

- Model: shallow water equations with horizontal bottom

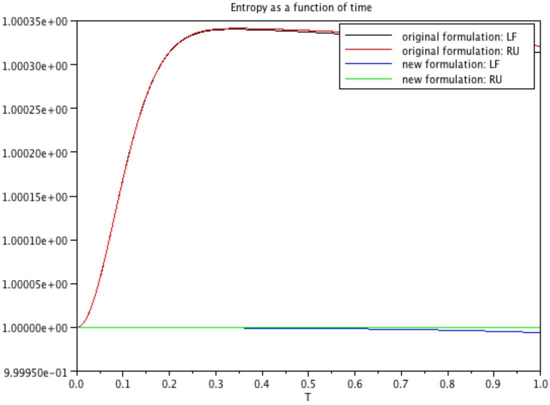
$$\partial_t h + \partial_x(hu) = 0, \quad \partial_t(hu) + \partial_x(hu^2 + g\frac{h^2}{2}) = \frac{\sigma}{\rho} h \partial_{xxx} h.$$

- Periodic boundary conditions

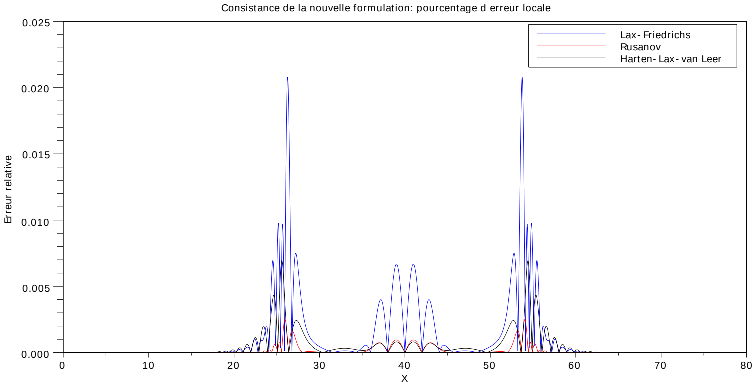


Entropy stability: numerical comparison II

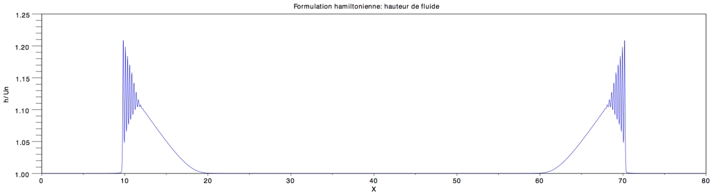
- Comparison of the original formulation and the "new" formulation
- Second order schemes for numerical simulations

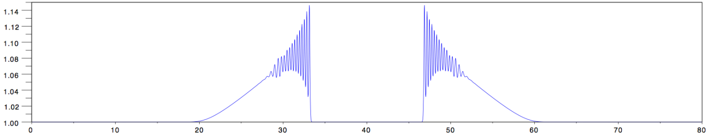


Schrödinger type formulation: numerical consistency



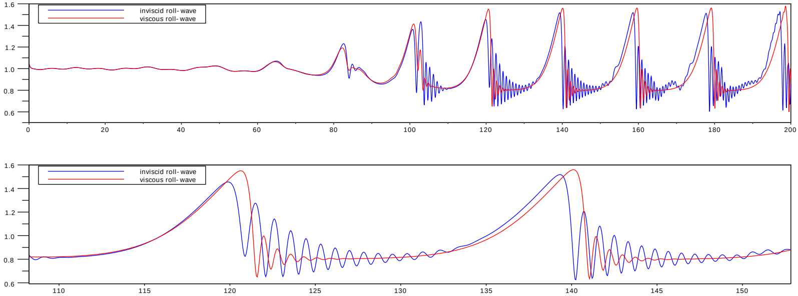
Hamiltonian formulation





- Conservative scheme in space: no numerical viscosity
- Time discretization: backward implicit Euler
- Formation of **dispersive shock waves**

Simulation of Liu Gollub experiment (Phys of Fluids 94)



- Numerical simulation for the shallow water model with $A_1 = 1$.
- Numerical scheme: Rusanov (2nd order) on the extended formulation.
- Reynolds number $Re = 29$, Inclination $\theta = 6.4^o$, Weber number $We = 35$.

Conclusion

1. Summary
   - Proof of entropy stability with a new form of Euler-Korteweg equations
   - Numerically: "new" formulation is more stable than original formulation
   - Hamiltonian semi-discretization.

2. Perspectives and Open problems
   - Generalization to 2d-motions (secondary instabilities)?
   - Higher order methods (Discontinous Galerkin methods)?

   - Hamiltonian semi-discretization: symplectic method for time discretization? (leap frog method/implicit-explicit schemes)
   - Other models: water wave models (Serre-Green/Naghdi)

$$\partial_t h + \partial_x(h\bar{u}) = 0,$$
$$\partial_t(h\bar{u}) + \partial_x\left(h\bar{u}^2 + p\right) = 0, \quad p = \frac{gh^2}{2} + \frac{h^2}{3}\ddot{h}.$$

   - Boundary conditions?