Admissibility and asymptotic-preserving scheme

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Outline

Outline

Problematic

Hyperbolic systems of conservation laws with source terms:

$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \tag{1}$$

- $\mathcal{A}$: set of admissible states,
- $\mathbf{W} \in \mathcal{A} \subset \mathbb{R}^N$,
- $\mathbf{F}$: physical flux,
- $\gamma > 0$: controls the stiffness,
- $\mathbf{R} : \mathcal{A} \to \mathcal{A}$: smooth function with some compatibility conditions (cf. [Berthon, LeFloch, and Turpault, 2013]).

Problematic

Hyperbolic systems of conservation laws with source terms:

$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \quad (1)$$

Under compatibility conditions on $\mathbf{R}$, when $\gamma t \to \infty$, (1) degenerates into a diffusion equation:

$$\partial_t w - \mathrm{div}\big(\mathbf{D}(w)\nabla w\big) = 0 \quad (2)$$

- $w \in \mathbb{R}$, linked to $\mathbf{W}$,
- $\mathbf{D}$: positive and definite matrix, or positive function.

Examples: isentropic Euler with friction

$$\left\{\begin{array}{rcl}\partial_t\rho+\partial_x\rho u+\partial_y\rho v & = & 0\\ \partial_t\rho u+\partial_x(\rho u^2+p(\rho))+\partial_y\rho uv & = & -\kappa\rho u\\ \partial_t\rho v+\partial_x\rho uv+\partial_y(\rho v^2+p(\rho)) & = & -\kappa\rho v\end{array}\right. \text{, with: } p'(\rho)>0,\ \kappa>0$$

$$\mathcal{A}=\{(\rho,\rho u,\rho v)^T\in\mathbb{R}^3/\rho>0\}$$

Formalism of (1)

- $\mathbf{W}=(\rho,\rho u,\rho v)^T$
- $\mathbf{R}(\mathbf{W})=(\rho,0,0)^T$
- $\mathbf{F}(\mathbf{W})=\begin{pmatrix}\rho u, & \rho u^2+p & ,\rho uv\\ \rho v, & \rho uv & ,\rho v^2+p\end{pmatrix}^T$
- $\gamma(\mathbf{W})=\kappa$

Limit diffusion equation

$$\partial_t\rho-\mathrm{div}\left(\frac{1}{\kappa}\nabla p(\rho)\right)=0$$

Examples: shallow water equations (linear friction)

$$\left\{\begin{array}{rcl}\partial_t h + \partial_x hu + \partial_y hv & = & 0 \\ \partial_t hu + \partial_x(hu^2 + gh^2/2) + \partial_y huv & = & -\frac{\kappa}{h^\eta} hu \\ \partial_t hv + \partial_x huv + \partial_y(hv^2 + gh^2/2) & = & -\frac{\kappa}{h^\eta} hv\end{array}\right. \text{, with: } \kappa > 0$$

$$\mathcal{A} = \{(h, hu, hv)^T \in \mathbb{R}^3 / h > 0\}$$

Formalism of (1)

- $\mathbf{W} = (h, hu, hv)^T$
- $\mathbf{R}(\mathbf{W}) = (h, 0, 0)^T$
- $\mathbf{F}(\mathbf{W}) = \begin{pmatrix} hu, & hu^2 + p & , huv \\ hv, & huv & , hv^2 + p \end{pmatrix}^T$
- $\gamma(\mathbf{W}) = \frac{\kappa}{h^\eta}$

Limit diffusion equation

$$\partial_t h - \mathrm{div}\left(\frac{h^\eta}{\kappa} \nabla \frac{gh^2}{2}\right) = 0$$

Examples: shallow water equations (Manning friction)

$$\left\{\begin{array}{rcl}\partial_t h + \partial_x hu + \partial_y hv &=& 0\\ \partial_t hu + \partial_x(hu^2 + gh^2/2) + \partial_y huv &=& -\frac{\kappa}{h^\eta}\|\mathbf{hu}\|hu\\ \partial_t hv + \partial_x huv + \partial_y(hv^2 + gh^2/2) &=& -\frac{\kappa}{h^\eta}\|\mathbf{hu}\|hv\end{array}\right.$$

$$\mathcal{A} = \{(h, hu, hv)^T \in \mathbb{R}^3 / h > 0\}$$

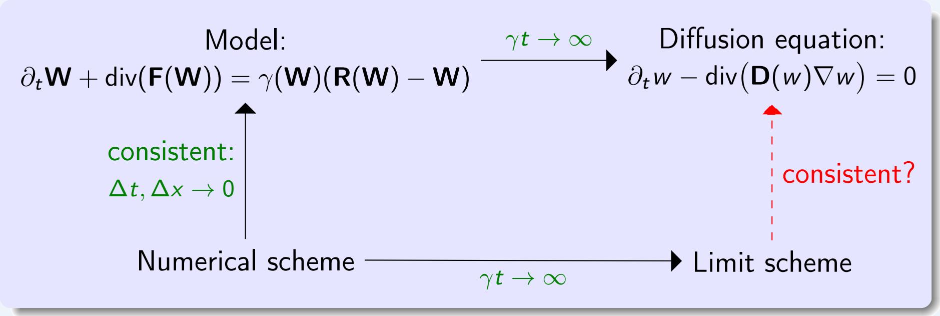
Formalism of (1)

- $\mathbf{W} = (h, hu, hv)^T$
- $\mathbf{R}(\mathbf{W}) = (h, 0, 0)^T$
- $\mathbf{F}(\mathbf{W}) = \begin{pmatrix} hu, & hu^2 + p & , huv \\ hv, & huv & , hv^2 + p \end{pmatrix}^T$
- $\gamma(\mathbf{W}) = \frac{\kappa}{h^\eta}\|\mathbf{hu}\|$

Limit equation

$$\partial_t h - \operatorname{div}\left(\sqrt{\frac{gh^\eta}{\kappa}}\frac{\sqrt{h}}{\sqrt{\|\nabla h\|}}\nabla h\right) = 0$$

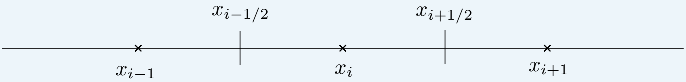
Aim of an AP scheme



Example of a non AP scheme in 1D

$$\partial_t \mathbf{W} + \partial_x(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \tag{1}$$

$$\begin{array}{rclrcl} \mathbf{W} & = & (\rho, \rho u)^T & \mathbf{F}(\mathbf{W}) & = & (\rho u, \rho u^2 + p)^T \\ \gamma(\mathbf{W}) & = & \kappa & \mathbf{R}(\mathbf{W}) & = & (\rho, 0)^T \end{array}$$



$$\frac{\mathbf{W}_i^{n+1} - \mathbf{W}_i^n}{\Delta t} = -\frac{1}{\Delta x}\left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}\right) + \gamma(\mathbf{W}_i^n)(\mathbf{R}(\mathbf{W}_i^n) - \mathbf{W}_i^n)$$

Limit

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{1}{2\Delta x^2}\left(b_{i+1/2}\Delta x(\rho_{i+1}^n - \rho_i^n) - b_{i-1/2}\Delta x(\rho_i^n - \rho_{i-1}^n)\right)$$

State-of-the-art for AP schemes in 1D

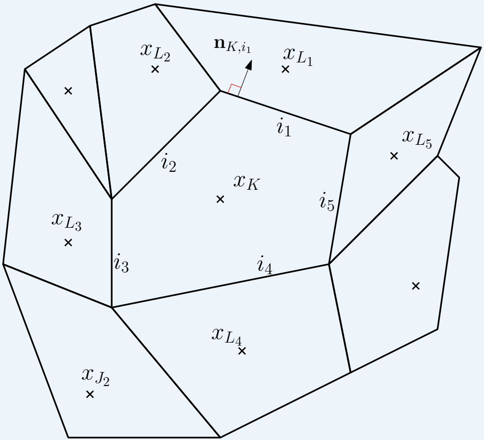
1. control of numerical diffusion:
   - telegraph equations: [Gosse and Toscani, 2002],
   - M1 model: [Buet and Després, 2006], [Buet and Cordier, 2007], [Berthon, Charrier, and Dubroca, 2007], ...
   - Euler with gravity and friction: [Chalons, Coquel, Godlewski, Raviart, and Seguin, 2010],
2. ideas of hydrostatic reconstruction used in ‘well-balanced’ scheme used to have AP properties:
   - Euler with friction: [Bouchut, Ounaissa, and Perthame, 2007],
3. using convergence speed and finite differences:
   - [Aregba-Driollet, Briani, and Natalini, 2012],
4. generalization of Gosse and Toscani:
   - [Berthon and Turpault, 2011],
   - [Berthon, LeFloch, and Turpault, 2013].

State-of-the-art for AP schemes in 2D

- Cartesian and admissible meshes $\Longrightarrow$ 1D
- unstructured meshes:
  1. MPFA based scheme:
     - [Buet, Després, and Franck, 2012],
  2. using the diamond scheme (Coudière, Vila, and Villedieu) for the limit scheme:
     - [Berthon, Moebs, Sarazin-Desbois, and Turpault, 2015],
  3. SW with Manning-type friction:
     - [Duran, Marche, Turpault, and Berthon, 2015].

Outline

Notations



- $\mathcal{M}$ is an unstructured mesh constituted of polygonal cells K,
- $\boldsymbol{x}_K$ is the center of the cell K,
- L is the neighbour of K by the interface i, $i = K \cap L$,
- $\mathcal{E}_K$ are the interfaces of K,
- $|e_i|$ is length of the interface i,
- $|K|$ is the area of the cell K,
- $\mathbf{n}_{K,i}$ is the normal vector to the interface i.

Aim of the development

- for any 2D unstructured meshes,
- for any system of conservation laws which could be written as (1),
- under a 'hyperbolic' CFL:
  - stability,
  - preservation of $\mathcal{A}$,
  - preservation of the asymptotic behaviour,

$$\max_{\substack{K \in \mathcal{M} \\ i \in \mathcal{E}_K}} \left(b_{K,i} \frac{\Delta t}{\Delta x} \right) \leq \frac{1}{2}.$$

Outline

Aim of an AP scheme

Model:
$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W})$$

$\gamma t \to \infty$

Diffusion equation:
$$\partial_t w - \mathrm{div}\big(\mathbf{D}(w)\nabla w\big) = 0$$

consistent:

$\Delta t, \Delta x \to 0$

consistent?

Numerical scheme

$\gamma t \to \infty$

Limit scheme

Choice of the limit scheme

FV scheme to discretize diffusion equations:

$$\partial_t w - \mathrm{div}(\mathbf{D}(w)\nabla w) = 0 \qquad (2)$$

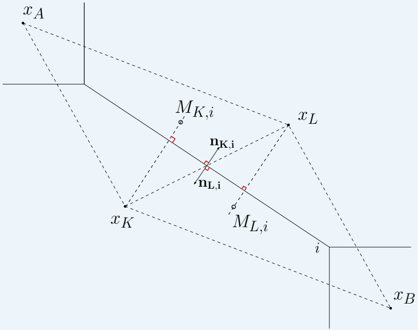
Choice: scheme developed by Droniou and Le Potier

- conservative and consistent,
- preserves $\mathcal{A}$,
- nonlinear.

$$(\mathbf{D} \cdot \nabla_i w_K) \cdot \mathbf{n}_{K,i} = \sum_{J \in S_{K,i}} \overline{\nu}^J_{K,i}(w)(w_J - w_K)$$

- $S_{K,i}$ the set of points used for the reconstruction on edges i of cell K
- $\overline{\nu}^J_{K,i}(w) \geq 0$

Presentation of the DLP scheme



$$
\begin{aligned}
M_{K,i} &= \sum_{J \in S_{K,i}} \omega_{K,i}^{J} \, X_J \qquad & w_{M_{K,i}} &= \sum_{J \in S_{K,i}} \omega_{K,i}^{J} \, w_J \\
M_{L,i} &= \sum_{J \in S_{L,i}} \omega_{L,i}^{J} \, X_J \qquad & w_{M_{L,i}} &= \sum_{J \in S_{L,i}} \omega_{L,i}^{J} \, w_J
\end{aligned}
$$

Presentation of the DLP scheme

Two approximations

$$\nabla_i w_K \cdot \mathbf{n}_{K,i} = \frac{w_{M_{K,i}} - w_K}{|KM_{K,i}|}$$
$$\nabla_i w_L \cdot \mathbf{n}_{L,i} = \frac{w_{M_{L,i}} - w_L}{|LM_{L,i}|}$$

Convex combination: $\gamma_{K,i} + \gamma_{L,i} = 1$, $\gamma_{K,i} \geq 0$, $\gamma_{L,i} \geq 0$

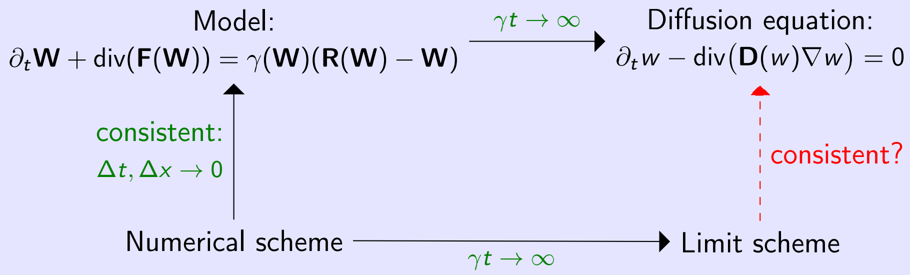
$$\nabla_i w_K \cdot \mathbf{n}_{K,i} = \gamma_{K,i}(w) \nabla_i w_K \cdot \mathbf{n}_{K,i} + \gamma_{L,i}(w) \nabla_i u_L \cdot \mathbf{n}_{L,i}$$
$$= \sum_{J \in S_{K,i}} \overline{\nu}^J_{K,i}(w)(w_J - w_K), \text{ with : } \overline{\nu}^J_{K,i}(w) \geq 0$$

Properties of the DLP scheme

- consistent with the diffusion equation on any mesh,
- satisfy the maximum principle.

Outline

Aim of an AP scheme



Scheme for the hyperbolic part

$$\mathbf{W}_K^{n+1} = \mathbf{W}_K^n - \frac{\Delta t}{|K|} \sum_{i \in \mathcal{E}_K} \boldsymbol{\mathcal{F}}_i(\mathbf{W}_K, \mathbf{W}_L, \dots) \cdot \mathbf{n}_{K,i} \qquad (3)$$

Theorem

We assume that the conservative flux $\boldsymbol{\mathcal{F}}_i$ has the following properties:

1. Consistency: if $\mathbf{W}_K^n \equiv \mathbf{W}$ then $\boldsymbol{\mathcal{F}}_i \cdot \mathbf{n}_{K,i} = \mathbf{F}(\mathbf{W}) \cdot \mathbf{n}_{K,i}$,
2. Admissibility:
   - a $\exists\ \nu_{K,i}^J \geq 0,\ \boldsymbol{\mathcal{F}}_i \cdot \mathbf{n}_{K,i} = \sum_{J \in \mathcal{S}_{\boldsymbol{K},\boldsymbol{i}}} \nu_{K,i}^J \boldsymbol{\mathcal{F}}_{KJ} \cdot \boldsymbol{\eta}_{KJ}$
   - b $\sum_{i \in \mathcal{E}_{\boldsymbol{K}}} |e_i| \sum_{J \in \mathcal{S}_{\boldsymbol{K},\boldsymbol{i}}} \nu_{K,i}^J \cdot \boldsymbol{\eta}_{KJ} = 0.$

Then the scheme (3) is stable, and preserves $\mathcal{A}$ under the classical following CFL condition:

$$\max_{\substack{\boldsymbol{K} \in \mathcal{M} \\ J \in \mathcal{E}_{\boldsymbol{K}}}} \left(b_{KJ} \frac{\Delta t}{\delta_J^K} \right) \leq 1. \qquad (4)$$

Example of fluxes

1. TP flux:

$$\mathcal{F}_i(\mathbf{W}_K, \mathbf{W}_L) \cdot \mathbf{n}_{K,i} = \frac{\mathbf{F}(\mathbf{W}_K) + \mathbf{F}(\mathbf{W}_L)}{2} \cdot \mathbf{n}_{K,i} - b_{KL}(\mathbf{W}_L - \mathbf{W}_K)$$

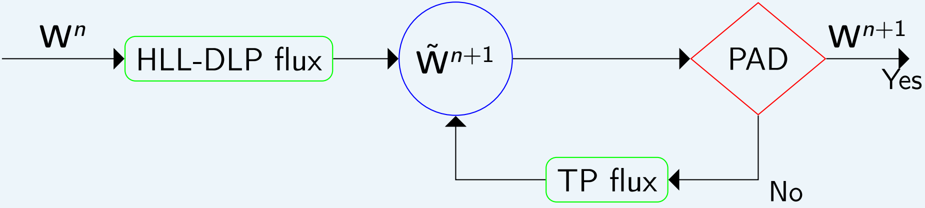
2. HLL-DLP flux:

$$\mathcal{F}_i(\mathbf{W}_K, \mathbf{W}_L, \mathbf{W}_J) \cdot \mathbf{n}_{K,i} = \sum_{J \in \mathcal{S}_{\boldsymbol{K},\boldsymbol{i}}} \nu_{K,i}^J \left(\frac{\mathbf{F}(\mathbf{W}_K) + \mathbf{F}(\mathbf{W}_J)}{2} \cdot \eta_{KJ} - b_{KJ}(\mathbf{W}_J - \mathbf{W}_K) \right)$$

But...

1. Fully respect the theorem, but not consistent in the diffusion limit
2. Does not respect the second property of admissibility, but consistent in the limit

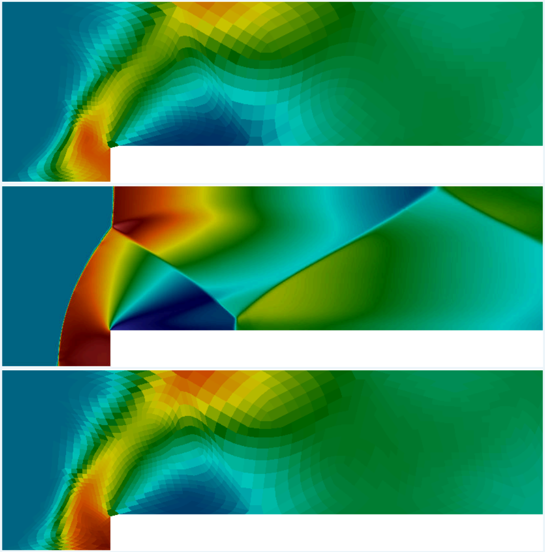
Procedure to preserve $\mathcal{A}$



1. $\tilde{\mathbf{W}}^{n+1}$ is computed with the HLL-DLP flux and with the CFL (4),
2. *Physical Admissiblility Detection* (PAD): if $\tilde{\mathbf{W}}^{n+1} \in \mathcal{A}$ then the time iterations can continue, else:
   - property 2b is enforced by using the TP flux on all not-admissible cells,
   - Δt and $\tilde{\mathbf{W}}^{n+1}$ are re-computed with the TP flux.

Outline

Wind tunnel with step



Outline

Scheme for the complete model

Complete system

$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) \tag{1}$$

$$\mathbf{W}_K^{n+1} = \mathbf{W}_K^n - \frac{\Delta t}{|K|} \sum_{i \in \mathcal{E}_K} |e_i| \overline{\mathcal{F}}_{K,i} \cdot \mathbf{n}_{K,i}, \tag{5}$$

Construction of $\overline{\mathcal{F}}_{K,i}$

$$\overline{\mathcal{F}}_{K,i} \cdot \mathbf{n}_{K,i} = \sum_{J \in \mathcal{S}_{K,i}} \nu_{K,i}^J \overline{\mathcal{F}}_{KJ} \cdot \boldsymbol{\eta}_{KJ}$$

$$\begin{aligned}\overline{\mathcal{F}}_{KJ} \cdot \boldsymbol{\eta}_{KJ} =& \alpha_{KJ} \mathcal{F}_{KJ} \cdot \boldsymbol{\eta}_{KJ} - (\alpha_{KJ} - \alpha_{KK}) \mathbf{F}(\mathbf{W}_K^n) \cdot \boldsymbol{\eta}_{KJ} \\ & - (1 - \alpha_{KJ}) b_{KJ} (\mathbf{R}(\mathbf{W}_K^n) - \mathbf{W}_K^n),\end{aligned}$$

Scheme for the complete model

- Is the scheme with the source term AP?

  $\Rightarrow$ generally not ...

Equivalent formulation

Rewrite (1) into:

$$\begin{aligned} \partial_t \mathbf{W} + \operatorname{div}(\mathbf{F}(\mathbf{W})) &= \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}), \qquad (1) \\ &= \gamma(\mathbf{W})(\mathbf{R}(\mathbf{W}) - \mathbf{W}) + (\overline{\gamma} - \overline{\gamma})\mathbf{W}, \\ \partial_t \mathbf{W} + \operatorname{div}(\mathbf{F}(\mathbf{W})) &= (\gamma(\mathbf{W}) + \overline{\gamma})(\bar{\mathbf{R}}(\mathbf{W}) - \mathbf{W}). \qquad (6) \end{aligned}$$

with:

- $\gamma(\mathbf{W}) + \overline{\gamma} > 0$
- $\bar{\mathbf{R}}(\mathbf{W}) = \frac{\gamma \mathbf{R}(\mathbf{W}) + \overline{\gamma}\mathbf{W}}{\gamma + \overline{\gamma}}$

Limit scheme

Introduction of $\overline{\gamma}$

$$\partial_t \mathbf{W} + \mathrm{div}(\mathbf{F}(\mathbf{W})) = (\gamma(\mathbf{W}) + \overline{\gamma}(\mathbf{W}))(\bar{\mathbf{R}}(\mathbf{W}) - \mathbf{W})$$

Rescaling

$$\begin{cases} \gamma & \leftarrow \dfrac{\gamma}{\varepsilon} \\ \Delta t & \leftarrow \dfrac{\Delta t}{\varepsilon} \end{cases}$$

$$h_K^{n+1} = h_K^n + \sum_{i \in \mathcal{E}_K} \frac{\Delta t}{|K|} |e_i| \sum_{J \in \mathcal{S}_{K,i}} \nu_{K,i}^{J,h} \frac{b_{KJ}^2}{2(\gamma_K + \bar{\gamma}_{K,i}^J)\delta_{KJ}^h} (h_J - h_K),$$

$$\Longrightarrow h_K^{n+1} = h_K^n + \sum_{i \in \mathcal{E}_K} \frac{\Delta t}{|K|} |e_i| \sum_{J \in \mathcal{S}_{K,i}} \overline{\nu}_{K,i}^J \frac{h^\eta}{\kappa_K} (g h_J^2/2 - g h_K^2/2).$$

Theorem

$$\mathbf{W}_K^{n+1} = \mathbf{W}_K^n - \frac{\Delta t}{|K|} \sum_{i \in \mathcal{E}_K} |e_i| \overline{\mathcal{F}}_{K,i} \cdot \mathbf{n}_{K,i} \tag{5}$$

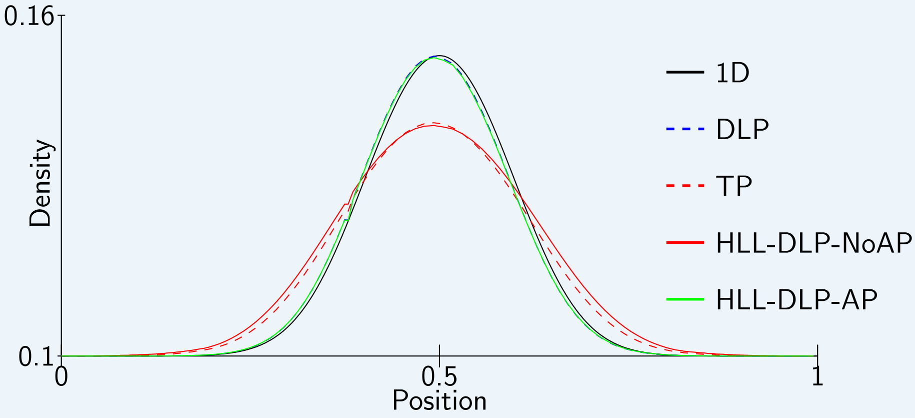
$$= \sum_{J \in \overline{\mathcal{E}_K}} \omega_{KJ} \left(\mathbf{W}_K^n - \frac{\Delta t}{\delta_J^K} [\tilde{\mathcal{F}}_{KJ} - \tilde{\mathcal{F}}_{KK}] \cdot \boldsymbol{\eta}_{KJ} \right)$$

The scheme (5) is consistent with the system of conservation laws, under the same assumptions of the previous theorem. Moreover, it preserves the set of admissible states $\mathcal{A}$ under the CFL condition:

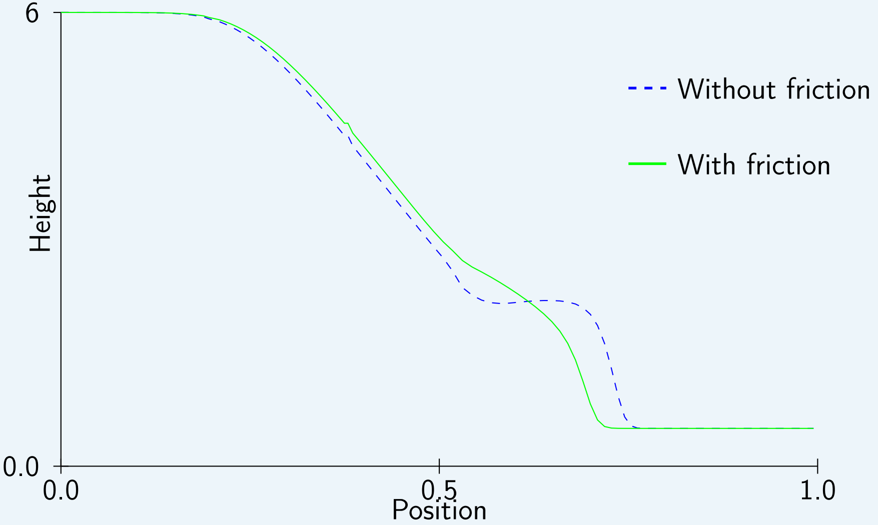
$$\max_{\substack{K \in \mathcal{M} \\ J \in \overline{\mathcal{E}_K}}} \left(b_{KJ} \frac{\Delta t}{\delta_J^K} \right) \leq 1. \tag{4}$$

Outline

Comparison in the diffusion limit



Dam break


6
0.0
Height
0.0
0.5
1.0
Position
Without friction
With friction

Outline

Conclusion and perspectives

Conclusion

- generic theory for various hyperbolic problems with asymptotic behaviours,
- first order scheme that preserve $\mathcal{A}$ and the asymptotic limit.

Perspectives

- extend the limit scheme to take care of diffusion systems and nonlinear diffusion equation,
- high-order schemes.

Thanks for your attention.

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