A bi-projection method for Bingham type flows

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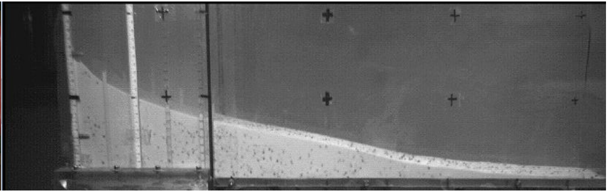
Summary

Complex fluids



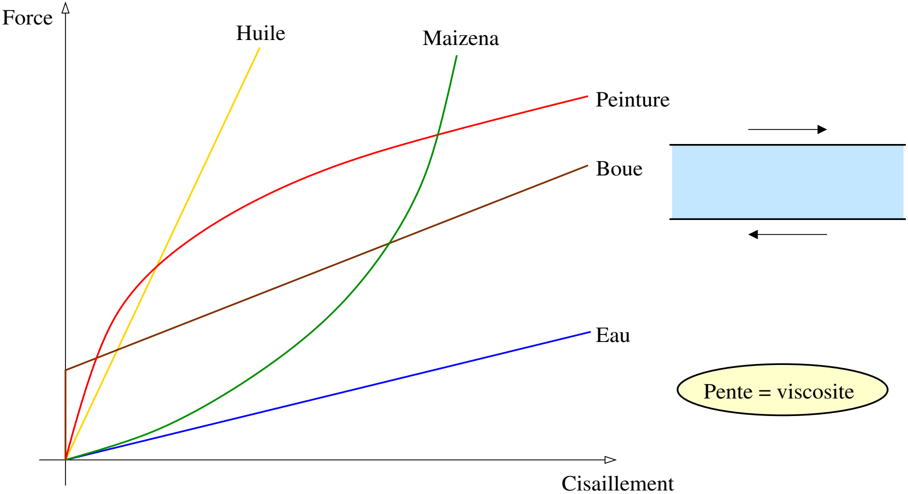




Rheology

Rheology is the study of the flow of matter, primarily in a liquid state, but also as solids under conditions in which they respond with plastic flow rather than deforming elastically in response to an applied force.



Summary

Bingham model

- Conservation equations (incompressible and isothermal case):

$$\begin{cases} \rho_0\big(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\big) + \nabla p = \operatorname{div} \boldsymbol{\tau}, \\ \operatorname{div} \mathbf{u} = 0. \end{cases}$$

- The stress is given by:

$$\boldsymbol{\tau} = 2\mu_0\,\mathrm{D}\mathbf{u} + \sigma_0 \frac{\mathrm{D}\mathbf{u}}{|\mathrm{D}\mathbf{u}|}$$

- The deformation tensor and its Froebenius norm are defined by:

$$\mathrm{D}\mathbf{u} = \frac{1}{2}\big(\nabla\mathbf{u} + {}^{T}(\nabla\mathbf{u})\big) \qquad \text{and} \qquad |\mathrm{D}\mathbf{u}|^2 = \sum_{1\le i,j\le d} |(\mathrm{D}\mathbf{u})_{ij}|^2.$$

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where the extra-stress satisfies:

$$\begin{aligned} \boldsymbol{\sigma} &= \frac{\mathrm{D}\mathbf{u}}{|\mathrm{D}\mathbf{u}|} && \text{if } |\mathrm{D}\mathbf{u}| \neq 0, \\ |\boldsymbol{\sigma}| &\leq 1 && \text{if } |\mathrm{D}\mathbf{u}| = 0. \end{aligned}$$

Classical approaches

1. Variational inequalities

$$\rho_0 \int_\Omega \left(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right)\cdot(\mathbf{v}-\mathbf{u}) + 2\mu_0 \int_\Omega \mathrm{D}\mathbf{u} : (\mathrm{D}\mathbf{v} - \mathrm{D}\mathbf{u}) + \sigma_0 \int_\Omega |\mathrm{D}\mathbf{v}| - |\mathrm{D}\mathbf{u}| \geq 0$$

✓ Well suited for finite volume methods

2. Regularization

$$\boldsymbol{\tau} = 2\mu_0\,\mathrm{D}\mathbf{u} + \sigma_0 \frac{\mathrm{D}\mathbf{u}}{|\mathrm{D}\mathbf{u}|} \quad \rightsquigarrow \quad \boldsymbol{\tau}_\varepsilon = 2\mu_0\,\mathrm{D}\mathbf{u}_\varepsilon + \sigma_0 \frac{\mathrm{D}\mathbf{u}_\varepsilon}{|\mathrm{D}\mathbf{u}_\varepsilon| + \varepsilon}$$

✓ Use results of the quasi-Newtonian models

✓ No stop in finite time!

Bingham model

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where the extra-stress satisfies:

$$\begin{matrix} \boldsymbol{\sigma} = \dfrac{\mathrm{D}\mathbf{u}}{|\mathrm{D}\mathbf{u}|} & \text{si } |\mathrm{D}\mathbf{u}| \neq 0, \\ |\boldsymbol{\sigma}| \leq 1 & \text{if } |\mathrm{D}\mathbf{u}| = 0. \end{matrix} \qquad \boldsymbol{\sigma} = \mathbb{P}(\boldsymbol{\sigma} + r\,\mathrm{D}\mathbf{u}),$$

where $\mathbb{P}$ is the projector on the following convex closed set:

$$\Lambda := \Big\{\boldsymbol{\sigma}\in L^2(\Omega)^{d\times d} \quad ; \quad |\boldsymbol{\sigma}(\mathbf{x})| \leq 1,\ \text{p.p.}\Big\}.$$

Bingham model

1. We introduce two non-dimensional numbers:

$$\mathfrak{Re} = \frac{\rho_0 V L}{\mu_0} \quad \text{and} \quad \mathfrak{Bi} = \frac{\sigma_0 L}{\mu_0 V}.$$

2. The model becomes

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \frac{1}{\mathfrak{Re}} \Delta \mathbf{u} = \frac{\mathfrak{Bi}}{\mathfrak{Re}} \operatorname{div} \boldsymbol{\sigma},$$

under the two constraints (r being any positif real)

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{and} \qquad \boldsymbol{\sigma} = \mathbb{P}(\boldsymbol{\sigma} + r\, \mathrm{D}\mathbf{u}).$$

Algorithm: time discretization

$\mathbf{u}^n$ and p^n being given we obtain $\widetilde{\mathbf{u}}^{n+1}$ solving

$$\begin{cases} \dfrac{\widetilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\delta t} + \nabla p^n - \dfrac{1}{\mathfrak{Re}} \Delta \widetilde{\mathbf{u}}^{n+1} = \mathbf{0} \\ \\ \widetilde{\mathbf{u}}^{n+1}\Big|_{\partial\Omega} = \mathbf{0}. \end{cases} \tag{1}$$

We obtain $(\mathbf{u}^{n+1}, p^{n+1})$ using the free divergence constraint:

$$\begin{cases} \dfrac{\mathbf{u}^{n+1} - \widetilde{\mathbf{u}}^{n+1}}{\delta t} + \nabla (p^{n+1} - p^n) = \mathbf{0} \\ \operatorname{div} \mathbf{u}^{n+1} = 0 \\ \mathbf{u}^{n+1} \cdot \mathbf{n}\Big|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

Algorithm: time discretization

$\mathbf{u}^n$, $\boldsymbol{\sigma}^n$ and p^n being given we obtain $\widetilde{\mathbf{u}}^{n+1}$, $\boldsymbol{\sigma}^{n+1}$ solving

$$\begin{cases} \dfrac{\widetilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\delta t} + \nabla p^n - \dfrac{1}{\mathfrak{Re}} \Delta \widetilde{\mathbf{u}}^{n+1} = \dfrac{\mathfrak{B}i}{\mathfrak{Re}} \operatorname{div} \boldsymbol{\sigma}^{n+1} \\ \boldsymbol{\sigma}^{n+1} = \mathbb{P}(\boldsymbol{\sigma}^{n+1} + r \, \mathrm{D}\widetilde{\mathbf{u}}^{n+1} + \theta(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^{n+1})) \\ \widetilde{\mathbf{u}}^{n+1}\Big|_{\partial\Omega} = \mathbf{0}. \end{cases} \tag{1}$$

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Algorithm: time discretization

Fixed point to solve (1) :

$$\begin{cases} \dfrac{\widetilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\delta t} + \nabla p^n - \dfrac{1}{\mathfrak{Re}} \Delta \widetilde{\mathbf{u}}^{n+1} = \dfrac{\mathfrak{B}i}{\mathfrak{Re}} \operatorname{div} \boldsymbol{\sigma}^{n+1} \\ \boldsymbol{\sigma}^{n+1} = \mathbb{P}(\boldsymbol{\sigma}^{n+1} + r\, \mathrm{D}\widetilde{\mathbf{u}}^{n+1} + \theta(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^{n+1})) \\ \widetilde{\mathbf{u}}^{n+1}\Big|_{\partial\Omega} = \mathbf{0}. \end{cases} \tag{1}$$

$\boldsymbol{\sigma}^{n,k}$ being given, we obtain $\widetilde{\mathbf{u}}^{n,k}$ then $\boldsymbol{\sigma}^{n,k+1}$:

$$\begin{cases} \dfrac{\widetilde{\mathbf{u}}^{n,k} - \mathbf{u}^n}{\delta t} + \nabla p^n - \dfrac{1}{\mathfrak{Re}} \Delta \widetilde{\mathbf{u}}^{n,k} = \dfrac{\mathfrak{B}i}{\mathfrak{Re}} \operatorname{div} \boldsymbol{\sigma}^{n,k}, \\ \boldsymbol{\sigma}^{n,k+1} = \mathbb{P}(\boldsymbol{\sigma}^{n,k} + r\, \mathrm{D}\widetilde{\mathbf{u}}^{n,k} + \theta(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^{n,k})), \\ \widetilde{\mathbf{u}}^{n,k}\Big|_{\partial\Omega} = \mathbf{0}. \end{cases} \tag{3}$$

Numerical study

Theorem 1

Assume that

$$2\theta + r\,\mathfrak{B}i \leq 2.$$

For each integer $n \in \mathbb{N}$, *the sequence* $(\widetilde{\mathbf{u}}^{n,k}, \boldsymbol{\sigma}^{n,k})_k$ *solution of system* (3) *converges to* $(\widetilde{\mathbf{u}}^{n+1}, \boldsymbol{\sigma}^{n+1})$, *solution of system* (1).

Moreover the convergence is geometric with common ratio $1 - \theta$.

Numerical study - Proof of theorem 1

1. Equation on differences $\overline{\mathbf{u}}^k = \widetilde{\mathbf{u}}^{n,k} - \widetilde{\mathbf{u}}^{n+1}$ and $\overline{\boldsymbol{\sigma}}^k = \boldsymbol{\sigma}^{n,k} - \boldsymbol{\sigma}^{n+1}$

$$\begin{cases} \frac{1}{\delta t}\overline{\mathbf{u}}^k - \frac{1}{\mathfrak{Re}}\Delta\overline{\mathbf{u}}^k = \frac{\mathfrak{B}i}{\mathfrak{Re}}\,\mathrm{div}\,\overline{\boldsymbol{\sigma}}^k, \\ \overline{\mathbf{u}}^k\Big|_{\partial\Omega} = 0, \\ \overline{\boldsymbol{\sigma}}^{k+1} = \mathbb{P}(\boldsymbol{\sigma}^{n,k} + r\,\mathrm{D}\widetilde{\mathbf{u}}^{n,k} + \theta(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^{n,k})) \\ \qquad\qquad - \mathbb{P}(\boldsymbol{\sigma}^{n+1} + r\,\mathrm{D}\widetilde{\mathbf{u}}^{n+1} + \theta(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^{n+1})). \end{cases}$$

2. Energy estimate:

$$\frac{1}{\delta t}\|\overline{\mathbf{u}}^k\|^2_{L^2(\Omega)} + \frac{1}{\mathfrak{Re}}\|\nabla\overline{\mathbf{u}}^k\|^2_{L^2(\Omega)} = -\frac{\mathfrak{B}i}{\mathfrak{Re}}\langle\overline{\boldsymbol{\sigma}}^k, \mathrm{D}\overline{\mathbf{u}}^k\rangle.$$

3. The projection $\mathbb{P}$ is a contraction:

$$\|\overline{\boldsymbol{\sigma}}^{k+1}\|^2_{L^2(\Omega)} \leq (1-\theta)^2\|\overline{\boldsymbol{\sigma}}^k\|^2_{L^2(\Omega)} + r^2\|\nabla\overline{\mathbf{u}}^k\|^2_{L^2(\Omega)} + 2r(1-\theta)\langle\overline{\boldsymbol{\sigma}}^k, \mathrm{D}\overline{\mathbf{u}}^k\rangle.$$

4. Simple combination gives the result.

Numerical study: stability result

Theorem 2 (Stability)

We assume that

$$r\,\mathfrak{B}i \leq 1 \quad et \quad \theta \leq 1/2$$

The sequence $(\mathbf{u}^n, \widetilde{\mathbf{u}}^n, p^n, \boldsymbol{\sigma}^n)_n$ *solution of system* (1)–(2) *is bounded.*

$$\begin{cases} \dfrac{\widetilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\delta t} + \nabla p^n - \dfrac{1}{\mathfrak{Re}} \Delta \widetilde{\mathbf{u}}^{n+1} = \dfrac{\mathfrak{B}i}{\mathfrak{Re}} \operatorname{div} \boldsymbol{\sigma}^{n+1} \\ \boldsymbol{\sigma}^{n+1} = \mathbb{P}(\boldsymbol{\sigma}^{n+1} + r\,\mathrm{D}\widetilde{\mathbf{u}}^{n+1} + \theta(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^{n+1})) \\ \dfrac{\mathbf{u}^{n+1} - \widetilde{\mathbf{u}}^{n+1}}{\delta t} + \nabla(p^{n+1} - p^n) = \mathbf{0} \\ \operatorname{div} \mathbf{u}^{n+1} = 0 \\ \mathbf{u}^{n+1} \cdot \mathbf{n}\Big|_{\partial\Omega} = 0 \quad \text{and} \quad \widetilde{\mathbf{u}}^{n+1}\Big|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Numerical study - Proof of theorem 2

1. Energy estimate:

$$\|\widetilde{\mathbf{u}}^{n+1}\|^2_{L^2(\Omega)} - \|\mathbf{u}^n\|^2_{L^2(\Omega)} + \|\widetilde{\mathbf{u}}^{n+1} - \mathbf{u}^n\|^2_{L^2(\Omega)} + \frac{2\delta t}{\mathfrak{Re}}\|\nabla\widetilde{\mathbf{u}}^{n+1}\|^2_{L^2(\Omega)}$$
$$= -2\delta t\langle\nabla p^n, \widetilde{\mathbf{u}}^{n+1}\rangle - \frac{2\delta t\,\mathfrak{Bi}}{\mathfrak{Re}}\langle\boldsymbol{\sigma}^{n+1}, \mathrm{D}\widetilde{\mathbf{u}}^{n+1}\rangle.$$

2. Control of $\langle\nabla p^n, \widetilde{\mathbf{u}}^{n+1}\rangle$ taking $\delta(\mathbf{u}^{n+1} + \widetilde{\mathbf{u}}^{n+1}) + \delta t^2\nabla(p^{n+1} + p^n)$ as test function:

$$\|\mathbf{u}^{n+1}\|^2_{L^2(\Omega)} + \delta t^2\|\nabla p^{n+1}\|^2_{L^2(\Omega)}$$
$$= 2\delta t\langle\nabla p^n, \widetilde{\mathbf{u}}^{n+1}\rangle + \|\widetilde{\mathbf{u}}^{n+1}\|^2_{L^2(\Omega)} + \delta t^2\|\nabla p^n\|^2_{L^2(\Omega)}.$$

3. Control of $\langle\boldsymbol{\sigma}^{n+1}, \mathrm{D}\widetilde{\mathbf{u}}^{n+1}\rangle$:

$$\theta\|\boldsymbol{\sigma}^{n+1}\|^2_{L^2(\Omega)} + (1-2\theta)\,\theta\|\boldsymbol{\sigma}^{n+1} - \boldsymbol{\sigma}^n\|^2_{L^2(\Omega)}$$
$$\leq 2\,r^2\|\nabla\widetilde{\mathbf{u}}^{n+1}\|^2_{L^2(\Omega)} + 2r\langle\boldsymbol{\sigma}^{n+1}, \mathrm{D}\widetilde{\mathbf{u}}^{n+1}\rangle + \theta\|\boldsymbol{\sigma}^n\|^2_{L^2(\Omega)}.$$

Numerical study: convergence result

Theorem 3 (Convergence)

We assume that

$$r\,\mathfrak{B}i \leq 1/3 \quad \textit{et} \quad \theta \leq 1/3$$

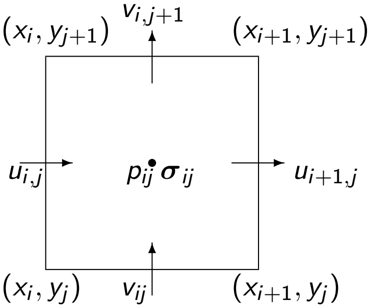
If there exists a regular solution $(\mathbf{u}, p, \boldsymbol{\sigma})$ *of continuous system then the sequence* $(\mathbf{u}^n)_{n\geq 0}$ *issued from the previous algorithm converges to* $\mathbf{u}$ *as* n *tends to* $+\infty$. *More precisely, there exists a constant* C *such that forall* $0 \leq n \leq N$, *we have*

$$\|\mathbf{u}(t_n) - \mathbf{u}^n\|_{L^2}^2 \;+\; \delta t \sum_{k=0}^{n} \|\nabla\mathbf{u}(t_k) - \nabla\mathbf{u}^k\|_{L^2}^2 \;\leq\; C\,(\theta\,\delta t + \delta t^2).$$

Summary

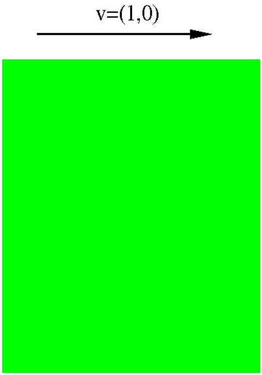
Numerical scheme and implementation

- *Time discretization*: second-order projection method based on the combination of the BDF2 (Backward Differentiation Formulae) and the AB2 (Adams-Bashforth) schemes.
- *Spatial discretization*: $\Omega = (0, L_x) \times (0, L_y)$ is discretized by using a Cartesian uniform mesh.

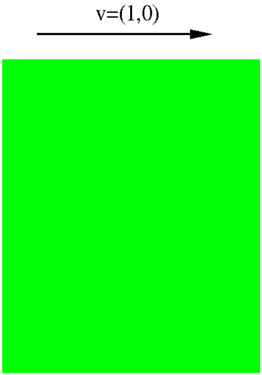


- *Implemented* in a F90/MPI code. The PETSc library to solve the linear systems and to manage data on structured grids.

Stationary flows in the lid-driven cavity at $\mathfrak{Re} = 1000$

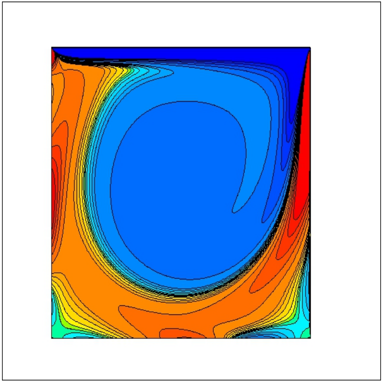


$\mathfrak{B}i = 0$



$\mathfrak{B}i = 10$

Stationary flows in the lid-driven cavity at $\mathfrak{R}e = 1000$



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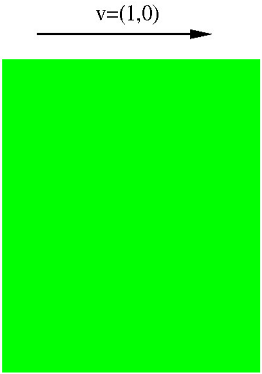

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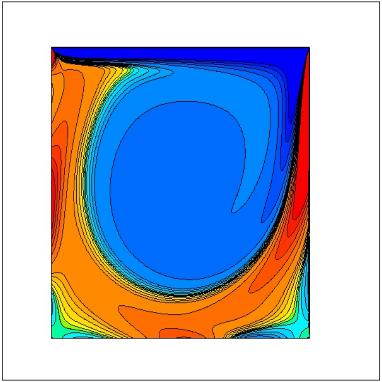
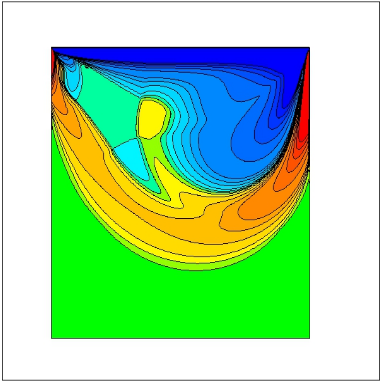


$\mathfrak{B}i = 0$



$\mathfrak{B}i = 10$

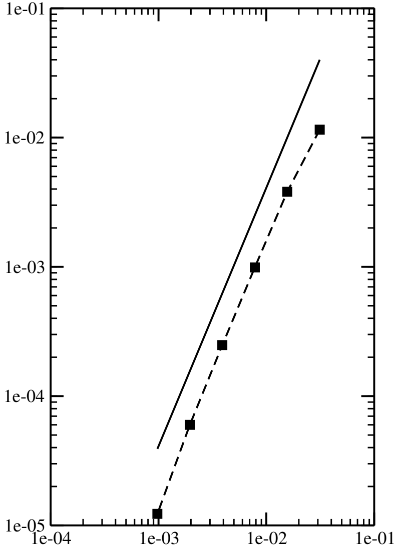
Numerical estimates of the convergence rate

Data:

- ✓ $\mathfrak{Re} = 1000$
- ✓ $\mathfrak{Bi} = 10$
- ✓ $r = 0.01$
- ✓ $\theta = \delta t$
- ✓ $N_x = N_y = 1024$

Error graph:

$$\|\mathbf{u} - \mathbf{u}_{\mathrm{ref}}\|_2 = F(\delta t)$$



Finite stopping times

Physical data:

✓ $\mathfrak{Re} = 1$

✓ $\mathfrak{Bi} = 1$

✓ Stopped training velocity at time $t = 0.05$

Numerical data:

✓ $r = 0.1$

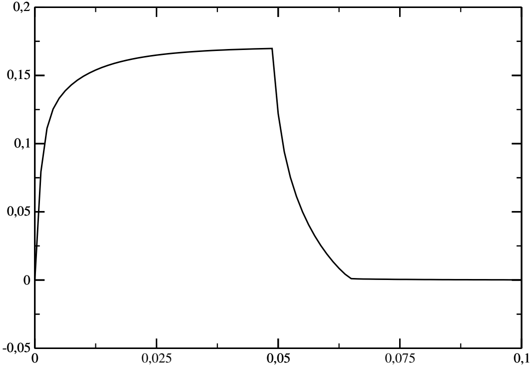
✓ $\theta = 0.00125$

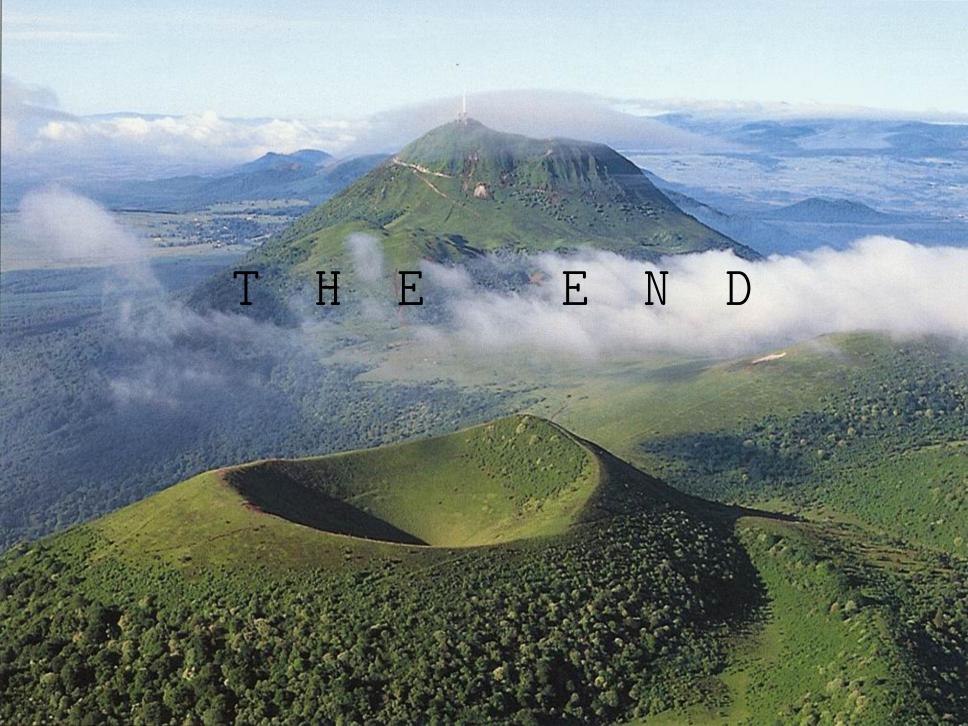
✓ $\delta t = 0.00125$

✓ $N_x = N_y = 128$

Graph of the L^2 norm:

$$\|\mathbf{u}\|_2 = G(t)$$




THE END