The effect of rough boundaries on laminar flows: a mathematical perspective

works with A. Basson, M. Bonnivard,

D. Bucur, A.L. Dalibard, E. Dormy, N. Masmoudi, M. Hillairet

David Gérard-Varet

Université Paris Diderot (Paris 7)

Part 1 : Navier-Stokes

Part 2 : Non-Newtonian/Rotating flows

Part 3 : Drag computation for rough solids close to contact

1. Setting of the problem

General concern: The effect of wall-roughness on fluid flows.

Two motivations for its study.

Motivation 1: computation of fluid flows

Pbs:

- Details of the roughness are unknown
- Too small for computational grids

Hope: to describe some averaged effect.

Idea: Replace the rough boundary by an artificial smooth one.
Prescribe there a homogenized boundary condition: *wall law*.

Question: What is the good wall law ?

Motivation 2: Microfluidics

Issue: To make fluids flow through very small devices.

Minimizing drag at the walls is welcome.

Many theoretical and experimental works.
[Tabeling, 2004], [Bocquet, 2007 and 2012], [Vinogradova, 2012].

Some of these works claim that the usual no-slip condition is not always satisfied at the micrometer scale:

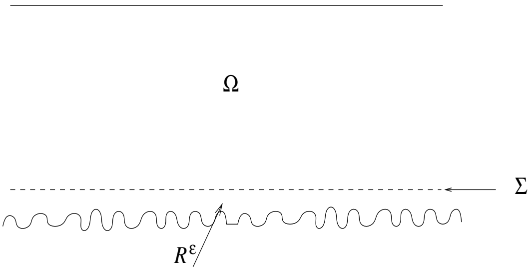
Some rough surfaces may generate a substantial slip.

However, these results are still debated . . .

... Maths may help, notably through a homogenization approach.

2. A simple model

2D rough channel: $\Omega^\varepsilon = \Omega \cup \Sigma \cup R^\varepsilon$



- Ω : smooth part: $\mathbb{R} \times (0,1)$.
- R^ε : rough part, *typical size* $\varepsilon \ll 1$.

$$R^\varepsilon \; = \; \{x = (x_1, x_2), \quad 0 > x_2 > \varepsilon\omega(x_1/\varepsilon)\}$$

ω with values in $(-1, 0)$, and K-Lipschitz.

- Σ : interface: $\mathbb{R} \times \{0\}$.

Stationary Navier-Stokes, with given flow rate:

$$\boxed{\begin{cases} u\cdot\nabla u - \Delta u + \nabla p \ = \ 0, \quad x \in \Omega^\varepsilon, \\ \operatorname{div} u = 0, \quad x \in \Omega^\varepsilon, \\ u|_{\partial\Omega^\varepsilon} = 0, \quad \displaystyle\int_\sigma u_1 = \phi, \end{cases}} \qquad (\mathrm{NS}^\varepsilon)$$

with $\phi > 0$, σ vertical cross-section.

Remark: Possible generalizations: 3D, unsteady flows.

Problem: Asymptotics $\varepsilon \to 0$.

Aim:

- To approximate u^ε by a solution of Navier-Stokes in Ω.
- To find the best *effective* (meaning regular in ε) boundary condition at Σ.

3. Asymptotics

a) Zeroth order approximation: Dirichlet boundary condition

Idea:
$$u^\varepsilon \approx u_D$$

where u_D is the solution of Navier-Stokes in Ω, with wall law

$$\boxed{u|_\Sigma = 0.}$$

Solution: *Poiseuille Flow* : $u_D = u_D(x_2) = \big(6\phi x_2(1 - x_2), 0\big)$.

Remark: Infinite channel : functions have infinite energy.

Theorem: *For ϕ and ε small enough,* $(\mathrm{NS}^\varepsilon)$ *has a unique solution u^ε in $H^1_{uloc}(\Omega^\varepsilon)$. Moreover,*

$$\|u^\varepsilon - u_D\|_{H^1_{uloc}(\Omega)} \le C\sqrt{\varepsilon},$$
$$\|u^\varepsilon - u_D\|_{L^2_{uloc}(\Omega)} \le C\,\varepsilon.$$

Remarks:

- Smallness of ϕ : natural for well-posedness.
- Requires only ω to be bounded and uniformly Lipschitz.
- Even well-posedness is not obvious. Lack of *a priori* bounds.

A typical sequence of approximations u_n^ε will satisfy

$$\int_{\Omega^\varepsilon} \dagger |\nabla u_n^\varepsilon|^2 = O(n) \xrightarrow[n\to+\infty]{} +\infty$$

Pb: To show that the energy does not concentrate.

Idea: [Ladyzenskaya et Solonnikov'83]

$$E_k := \int_{\Omega_k^\varepsilon} |\nabla u_n^\varepsilon|^2, \quad \Omega_k^\varepsilon \ := \ \Omega^\varepsilon \cap \{|x_1| \leq k\}.$$

One shows by induction on $n-k$ that $E_k = O(k)$ for all $k < n$.

Possible here thanks to the induction relation

$$E_k \leq C\left(E_{k+1} - E_k) + (E_{k+1} - E_k)^{3/2} + k + 1\right).$$

Simpler example:

$$-\Delta u^\varepsilon = 1 \quad \text{in } \Omega^\varepsilon, \quad u|_{\partial\Omega^\varepsilon} = 0.$$

Multiply by $\chi_k u^\varepsilon$, with $\chi_k = 1$ over Ω^ε_k, integrate:

$$\int_{\Omega^\varepsilon} \chi_k |\nabla u^\varepsilon|^2 \leq \int_{\Omega^\varepsilon} \nabla\chi^k \cdot \nabla u^\varepsilon u^\varepsilon + \int_{\Omega^\varepsilon} \chi^k u^\varepsilon.$$

Then :

$$\int_{\Omega^\varepsilon} \chi_k |\nabla u^\varepsilon|^2 \leq C\left(\int_{\Omega^\varepsilon_{k+1}\setminus\Omega^\varepsilon_k} |\nabla u^\varepsilon|^2 + \int_{\Omega^\varepsilon_{k+1}\setminus\Omega^\varepsilon_k} |u^\varepsilon|^2 + k + 1\right).$$

Crucial ingredient: Poincaré's inequality in a channel.

We find: $E_k \leq C((E_{k+1} - E_k) + k + 1)$.

For Navier-Stokes:

- The term $(E_{k+1} - E_k)^{3/2}$ comes from the nonlinearity.
- The pressure term must be treated carefully.

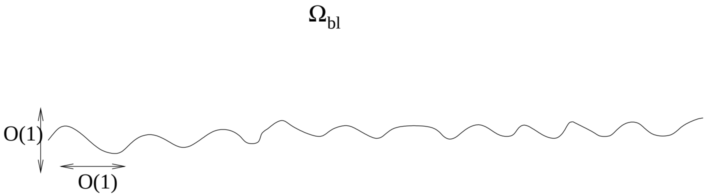
Conclusion: The no-slip condition provides a $O(\varepsilon)$ approx. in L^2.
Can we find a better one ?

b) First order approximation: Navier boundary condition

Two ideas behind this slip.

Idea 1:
$$u^\varepsilon \approx u_D + 6\phi\varepsilon v\left(\frac{x}{\varepsilon}\right),$$

$v = v(y)$: *Boundary layer corrector*. Cancels the trace of u_D at Γ^ε.



Defined on $\Omega^{bl} := \{y_2 > \omega(y_1)\}$. Formally,

$$\boxed{\begin{cases} -\Delta v + \nabla p = 0, & y \in \Omega^{bl}, \\ \operatorname{div} v = 0, & y \in \Omega^{bl}, \\ v(y) = (-\omega(y_1), 0), & y \in \partial\Omega^{bl}. \end{cases}} \qquad \text{(BL)}$$

Idea 2: The boundary layer generates a non-zero mean flow

$$v \to v^\infty = (\alpha, 0), \qquad \text{as } y_2 \to +\infty, \text{ for some } \alpha > 0.$$

Consequence: Formal expansion yields

$$u^\varepsilon \approx u_D + 6\phi\varepsilon(\alpha, 0) + o(\varepsilon) \quad \text{in } L^2$$

A better approximation should be the solution u_N of NS in Ω with *Navier boundary condition*:

$$\boxed{u_2|_\Sigma = 0, \quad u_1|_\Sigma = \varepsilon\,\alpha\,\partial_2 u_1|_\Sigma.}$$

Pb: To make these formal ideas rigorous !

The analysis of system (BL) *is difficult.*

- *Well-posedness*:
  No tangential decay at infinity. Requires local bounds.
  No Poincaré's inequality.
  No maximum principle, no Harnack's inequality.

- *Behaviour as $y_2 \to +\infty$?*

One easier setting: periodic roughness. [Achdou et al, Jäger et al]

- Solvability: Variational formulation in a space of functions periodic with respect to y_1.
- $y_2 \to +\infty$: Fourier series in y_1. *Convergence at exponential rate* of v to $(\alpha, 0)$,

$$\alpha = L^{-1} \int_0^L v_1(y_1, 0)\, dy_1.$$

General setting: much harder.

Still: *Well-posedness holds for general* ω*.*

Theorem: *System* (BL) *has a unique solution* $v \in H^1_{loc}(\overline{\Omega^{bl}})$ *satisfying*

$$\sup_{k \in \mathbb{Z}} \int_{\Omega^{bl}_{k,k+1}} |\nabla v|^2 < +\infty,$$

where $\Omega^{bl}_{k,k+1} := \Omega^{bl} \cap \{k \le y_1 \le k+1\}$.

Proof: *Inspired by transparent boundary conditions in numerical analysis.*

Idea 1: To restrict system (BL) to the lower part of Ω^{bl}

$$\Omega^{bl}- \ := \ \Omega^{bl} \cap \{y_2 < 0\}.$$

Pb: What condition at the upper boundary $y_2 = 0$?

Formally: $-\Delta v + \nabla q = 0$ in the half-plane $y_2 > 0$.
Fourier transform in y_1. Solve the ODE in y_2.

The condition at $y_2 = 0$ is given by

$$\boxed{(\partial_2 v - q e_2)\,|_{y_2=0} \ = \ DN\,(v|_{y_2=0})}$$

where DN is a Dirichlet to Neumann operator defined formally by

$$\mathcal{F} DN(v_0)(\xi) \ := \ \begin{pmatrix} -2|\xi| & -i\xi \\ i\xi & -2|\xi| \end{pmatrix} \mathcal{F} v_0(\xi).$$

Idea 2: The domain Ω_{-}^{bl} is a bounded channel. Methods used in Theorem 1 can apply.

Difficulties:

- To extend the DN operator to $H^{1/2}_{uloc}(\mathbb{R})$.
- To justify the equivalence between the original system and the new one.
- To prove the induction on the truncated energies E_k despite the non-local character of DN.

Question: Asymptotic behavior ? Does $v \to v^{\infty}$ as $y_2 \to +\infty$?

Claim: Very unlikely to be true.

Dirichlet problem: $\boxed{\Delta v = 0 \quad \text{in } y_2 > 0, \quad v|_{y_2=0} = v_0}$.

- If v_0 1-periodic, then $v(0, y_2) \to \int_0^1 v_0$ exponentially fast.
- *There exists $v_0 \in L^\infty(\mathbb{R})$ such that $v(0, y_2)$ has no limit.*

Take $v_0 = (-1)^k$ in $[a^k, a^{k+1}]$, $y_2 = 2^n$, and use the formula

$$v(0, y_2) \;=\; \frac{1}{\pi} \int_{\mathbb{R}} \frac{y_2}{y_2^2 + t^2} v_0(t)\, dt.$$

Remark: v_0 with values in $\{+1, -1\}$: *close to coin tossing*.

Suggests random modelling of the roughness.

c) Random roughness.

Realistic modelling: *Roughness randomly distributed, following a stationary process.*

Basically: We endow the set of all possible boundaries

$$P \;=\; \{\omega \text{ with values in } (-1,0), \text{ K-lip}\}$$

with the cylindrical σ-field, and a probability measure μ.

Stationarity: μ *is invariant under the group of translations*

$$\tau_h : P \mapsto P, \quad \omega \mapsto \omega(\cdot + h).$$

Domains Ω^ε, Ω^{bl}, functions u^ε, v ...depend on ω.

Theorem 3: *There exists* $\alpha = \alpha(\omega) \in L^2(P)$ *such that:*

$$\|u^\varepsilon - u_N\|_{L^2_{uloc}(P\times\Omega)} = o(\varepsilon)$$

with

$$\|f\|^2_{L^2_{uloc}(P\times\Omega)} := \sup_t \mathbb{E} \int_{\Omega\cap\{|x_1-t|<1\}} |f|^2 dx\, d\mu$$

Remark: α explicit, linked to (BL). If μ is ergodic, α does not depend on ω.

Proof:: Keypoint is to show that $v \to (\alpha, 0)$ as $y_2 \to +\infty$.

Idea 1: *Use of Stokes double layer potential*: for all $y_2 > 0$,

$$\boxed{v(\omega, y) \ = G(\cdot, y_2) * v|_{y_2=0}(y_1)},$$

$$G(y) = \frac{2y_2}{\pi\left(y_1^2+y_2^2\right)^2} \begin{pmatrix} y_1^2 & y_1 y_2 \\ y_1 y_2 & y_2^2 \end{pmatrix}.$$

Idea 2: *Ergodic theorem:*

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R v(\omega, y_1 - h, 0)dh = v^{\infty}(\omega) = (\alpha(\omega), 0)$$

Convergence a.s., and in $L^p(P)$ with finite p, uniformly local in y_1.

If μ is ergodic, v^{∞} is constant.

One concludes through integration by parts in the integral formulation.

Summary:
Dirichlet's wall law: $O(\varepsilon)$ approx.
Navier's wall law: $o(\varepsilon)$ approx. Can we say more ?

Remark: The integral formula for v involves a family of mappings indexed by y_2:

$$S^{y_2} : v^0 \;\mapsto\; G(\cdot, y_2) * v^0$$

Defines (at a formal level) a semi-group. *Behaviour as $y_2 \to +\infty$ is linked to the spectral properties of S^{y_2}.*

Periodic roughness: $v^0 = v^0(y_1)$ is periodic.

S^{y_2} contraction in $L^2(\mathbb{T})$. Fourier in y_1:

- 1 simple eigenvalue associated to constant functions.
- *Spectral gap*, hence convergence at exponential rate.

Hence, $\varepsilon \| v(x/\varepsilon) - v^\infty \| \;=\; O(\varepsilon^{3/2}) \;\longrightarrow$ Navier's law: $O(\varepsilon^{3/2})$.

Stationary roughness:

$$v^0 = v^0(\omega, y_1) \; = \; v^0(\tau_{y_1}(\omega), 0).$$

S^{y_2} contraction in $L^2(P)$. Spectrum can be more complicated.

Ergodic th : $\quad \varepsilon \| v(x/\varepsilon) - v^\infty \| \; = o(\varepsilon). \; \longrightarrow$ Navier's law: $o(\varepsilon)$.

Questions: Speed of convergence for v ? Csq on Navier law

Formally, spectrum related to the spectrum of the *shift*

$$L^2(P) \; \mapsto \; L^2(P), \quad V \; \mapsto \; V \circ \tau_h.$$

If τ_h is mixing, this operator has continuous spectrum.

Problem: To quantify the dispersion created by the continuous spectrum.

Tool: Central limit theorem.

Remark: Analogy with coin tossing.

Aim: To quantify the speed of convergence of

$$\frac{1}{N}\int_0^N v(\omega, h, 0)\, dh \;=\; \frac{1}{N}\sum_{k=0}^{n-1} X^k(\omega)$$

with $X^k = \int_k^{k+1} v(\omega, h, 0)\, dh$.

If the random variables X^k were independent: *Central limit theorem*.

Decay of correlations : In brief, *if correlations between X_k and X_l decay fast enough as $|k-l| \to \infty$, the central limit theorem is still valid.*

This suggest the following assumption on the roughness distribution:

(H): *Independence at large distances:*

$$\sigma\left(y_1 \mapsto \omega(y_1),\, y_1 \le a\right) \text{ and } \sigma\left(y_1 \mapsto \omega(y_1),\, y_1 \ge b\right)$$

are independent for $b-a$ small enough.

Remark: Far from the periodic case.

Another technical assumption:

(H'): *Measure μ has support $P_\alpha = \{\omega \in P, \quad \|\omega\|_{C^{2,\alpha}} \le K_\alpha\}$.*

Theorem: *Under assuptions (H), (H'),* $\frac{1}{N}\int_0^N v(\omega, h, 0)\, dh$ *satisfies a central limit theorem.*

Theorem':

$$\sqrt{y_2}\, \|v(\cdot,\cdot,y_2) - \alpha\|_{L^2_{uloc}(P\times\mathbb{R})} \xrightarrow[y_2\to+\infty]{} \sigma \geq 0.$$

$\to$ Navier's wall law: $O(\varepsilon^{3/2}|\ln\varepsilon|^{1/2})$.

Idea of the proof: To show that (H) implies a good decay of correlations for the spatial process $v(\omega, y_1, 0)$.

In brief, resumes to the following problem:

Show that if $\omega_1 = \omega_2$ *on* $[-n, n]$*, then the corresponding solutions of* (BL) *satisfy for some* $\alpha > 1/2$.

True with $\alpha = 1$!

Difficulties: Not defined on the same domain, estimate at a single point.

Idea: *Estimate on the Green function* $G_\omega(z, y)$, satisfying

$$\begin{cases} -\Delta G_\omega(z,\cdot) + \nabla P_\omega(z,\cdot) = \delta_z I_2, & y_2 > \omega(y_1), \\ G_\omega(z,\cdot) = 0, & y_2 = \omega(y_1). \end{cases}$$

coupled to the formula

$$v(\omega, 0, 0) \ = \ \int_{\{y_2=0\}} G_\omega(0, y) e_1 \, dy$$

Key estimate: For all z, y s.t. $|z - y| \geq 1$,

$$|\nabla_y G_\omega(z, y)| \ \leq \ C \, \frac{\delta(z)(1 + \delta(y))}{|z - y|^2}.$$

where δ is the distance to the boundary.

Remark: *For large values of* $|z-y|$ *, the oscillating boundary can be seen as low amplitude and high frequency.* $(\varepsilon = |z-y|^{-1})$.

Requires refined regularity estimates for Stokes, in

$$D^\varepsilon(0,1) \;:=\; D(0,1) \cap \{x_2 > \varepsilon\omega(x_1/\varepsilon)\}$$

If u satisfies

$$\begin{cases} -\Delta u + \nabla p = \operatorname{div} f, & x \in D^\varepsilon(0,1) \\ \operatorname{div} u = 0, & x \in D^\varepsilon(0,1) \\ u = 0, & x \in \Gamma^\varepsilon(0,1) \end{cases}$$

$$\text{then } \|\nabla u\|_{L^\infty(D^\varepsilon(0,1/2))} \leq \; C\left(\|u\|_{L^2(D^\varepsilon(0,1))} + \|f\|_{C^{0,\nu}(D^\varepsilon(0,1))}\right)$$

Inspired by works of Avellaneda and Lin on the homogenization of elliptic operators with periodic coefficients.

4. Real or apparent slip ?

Summary: Rigorous derivation of a Navier condition at Σ.

Question: Does it prove that roughness enhances slip ?

Not clear ! The positivity of α is linked to the position of our artificial boundary (namely *above the humps*).

If we keep the artificial boundary at $x_2 = 0$ and shift the roughness, things change.

Example: periodic roughness. One shows [Achdou et al, Jäger et al]

$$\alpha(\omega + h) \; = \; \alpha(\omega) - h, \quad \forall h,$$
$$\sup -\omega \; \le \; \alpha(\omega) \; \le \; \inf -\omega.$$

In our setting : $\omega < 0$, so $\alpha > 0$.

Only meaningful case: $<\omega>=0$: same averaged flow rate in the rough and smooth channels.

Problem: Find the maximizer and maximum of

$$\tilde{\alpha}(\omega) \; := \; \alpha(\omega) - \; <\omega>$$

among all rough profiles $\omega \in W^{1,\infty}(\mathbb{T})$ ($W^{1,\infty}(\mathbb{T}^2)$ in 3d).

Proposition: Maximum slip coefficient is achieved for flat surfaces:

$$\max_{\omega} \tilde{\alpha}(\omega) = \tilde{\alpha}(0) = 0.$$

Conclusion: apparent slip, not real.

5. Back to microfluidics

Question: May rough surfaces generate significant slip ?

Preliminary mathematical question:

Is there a "microscopic" condition at $\partial\Omega^\varepsilon$ that can give rise to "macroscopic" slip at $\partial\Omega$?

Intuition: Yes, at least if we consider some *pure slip* at $\partial\Omega^\varepsilon$:

$$\boxed{u \cdot \nu^\varepsilon|_{\partial\Omega^\varepsilon} = 0, \quad D(u)\nu^\varepsilon \times \nu^\varepsilon|_{\partial\Omega^\varepsilon} = 0.} \qquad \text{(S)}$$

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Answer: No, as soon as the roughness is non-degenerate !

See [Casado-Diaz et al, 03], [Bucur et al, 08]

.

Broadly, under the assumption

(A) The Young measures μ_y *(*$y \in \mathbb{R}$*) associated to the sequence* $(\omega'(\cdot/\varepsilon))$ *have a non-trivial support for a.e.* y*,*

any weak accumulation point u of a sequence of solutions (u^ε) in $H^1_{loc}(\Omega)$ will satisfy $u|_{\partial\Omega} = 0$.

Example: If ω is periodic and non-cst, $u|_{\partial\Omega} = 0$.

Formal idea:

Vanishing of the normal component $+$ high frequency oscillations of the boundary $+$ bound on ∇u^ε

$\longrightarrow$ vanishing of the whole velocity as $\varepsilon \to 0$.

One can be more quantitative, under a slightly different assumption:

(A') There is $C > 0$, such that for all $u \in C_c^\infty(\overline{R})$,

$$\boxed{u \cdot \nu|_{\{y_2=\gamma(y_1)\}} = 0 \Rightarrow \|u\|_{L^2(R)} \le C \|\nabla u\|_{L^2(R)}}$$

Theorem: *There exists $\phi_0 > 0$ such that for all $\phi < \phi_0$, $\varepsilon \le 1$, system* (NS$^\varepsilon$)-(S) *has a unique solution $u^\varepsilon \in H^1_{uloc}(\Omega^\varepsilon)$.*

Moreover, if (A') holds,

$$\boxed{\|u^\varepsilon - u\|_{H^1_{uloc}(\Omega)} \le C\phi\sqrt{\varepsilon}, \quad \|u^\varepsilon - u\|_{L^2_{uloc}(\Omega)} \le C\phi\varepsilon,}$$

where u is the Poiseuille flow in Ω (that satisfies $u|_{\partial\Omega} = 0$).

Remarks

1. The theorem shows that the effective slip can not be more than $O(\varepsilon)$.

   Boundary layer analysis: under ergodicity properties of ω, one shows that the effective slip is indeed $O(\varepsilon)$.

2. Assumption (A'):

   Amounts to (A) for periodic or quasiperiodic roughness: it is satisfied by non-cst boundary.

   Stationary ergodic case: (A') seems stronger than (A).

Conclusion: suggests that roughness is far from enhancing slip !

But still:

One can argue that our isotropic scaling for the roughness is very peculiar ...

To analyse more general scalings would be good.

Closer look at some physics papers:

- Rough (hydrophobic) surfaces generate bubbles in their hollows:
- The fluid slips above hollows, sticks at bumps.

Suggestion: To consider a model with a *flat boundary, alternating zones of slip and no-slip*, with arbitrary relative areas.

Example: $\Omega = \mathbb{T}^2 \times \mathbb{R}_+$ (3d model).

- Stokes in Ω, with some forcing.
- Boundary $\mathbb{T}^2 \times \{0\}$ divided in $\sim \varepsilon^{-2}$ square cells of side ε:

$$C_k^\varepsilon \; := \; \varepsilon\,(k \; + \; C), \quad C = [0,1[^2, \quad k \in [[0, \varepsilon^{-1} - 1]]^2$$

  with patches

$$P_k^\varepsilon \; = \; \varepsilon(k \; + \; P^\varepsilon), \quad P^\varepsilon \subset C.$$

- B.C. is *pure slip at* $\cup(C_k^\varepsilon \setminus P_k^\varepsilon)$, *no-slip at* $\cup P_k^\varepsilon$,

Question : Averaged boundary condition as $\varepsilon \to 0$?

Key: Volume fraction of no-slip: $\phi^\varepsilon = |P^\varepsilon| \in [0,1]$.

Two main results:

1. One for *patches*: broadly, $P^\varepsilon \Subset C$ smooth open set.
2. One for *riblets*: $P^\varepsilon = [0,1] \times I^\varepsilon$, I^ε subinterval.

"Theorem for patches"

- *If* $\phi^\varepsilon >> \varepsilon^2$, *the limit condition is Dirichlet.*
- *If* $\phi^\varepsilon << \varepsilon^2$, *the limit condition is pure slip.*
- *If* $\phi^\varepsilon \sim \varepsilon^2$, *the limit condition is Navier.*

"Theorem for riblets": $C > 0$ arbitrary.

- *If* $\phi^\varepsilon >> \exp(-C/\varepsilon)$, *the limit condition is Dirichlet.*
- *If* $\phi^\varepsilon << \exp(-C\varepsilon)$, *the limit condition is pure slip.*
- *If* $\phi^\varepsilon \sim \exp(-C/\varepsilon)$, *the limit condition is Navier.*

Remarks:

- Significant slip is possible. But the relative area of the no-slip zone needs to be very small (unrealistic ?).
- The riblet geometry is less efficient in improving slip.

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- Significant slip is possible. But the relative area of the no-slip zone needs to be very small (unrealistic ?).
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Proof: More or less already done ! Think of the simpler problem:

$$\boxed{\Delta u^\varepsilon = 0 \text{ in } \Omega, \quad \partial_\nu u^\varepsilon = 1 \text{ in } \cup (C_k \setminus P_k^\varepsilon), \quad u^\varepsilon = 0 \text{ in } \cup P_k^\varepsilon.}$$

Homogenization of the fractional Laplacian in domains with holes.

Allows to connect to the existing litterature [Cioranescu et al, 82], [Allaire, 91], [Caffarelli-Mellet, 08].