The effect of rough boundaries on laminar flows: a mathematical perspective

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Part 1 : Navier-Stokes

Part 2 : Non-Newtonian/Rotating flows

Part 3 : Drag computation for rough solids close to contact

Last talk: Navier-Stokes flow in a laminar regime.

Linear model for the boundary layer due to roughness.

Question: Examples of nonlinear models for the boundary layer ?

First example: Non-newtonian flows, with power law (work with A. Wroblewska).

$$\boxed{\begin{cases} -\mathrm{div}\, S(Du) + \nabla p = e_1 & \text{in } \Omega^\varepsilon, \\ \mathrm{div}\, u = 0 & \text{in } \Omega^\varepsilon, \\ u|_{\Gamma^\varepsilon} = 0, \quad u|_{x_2=1} = 0. \end{cases}} \qquad \text{(NN)}$$

where $\boxed{S(A) = \nu |A|^{p-2} A}$.

Interesting case: $1 < p \le 2$ ($p = 2$: newtonian).

For simplicity: *periodicity of the roughness profile* ω.

Again, the limit of u^ε is u^0, satisfying Dirichlet at the artificial boundary.

Modified Poiseuille flow: $\boxed{u^0(x) = (U(x_2), 0)}$ with

$$\boxed{U(x_2) = \frac{p-1}{p}\left(\sqrt{2}^{-\frac{p}{(p-1)}} - \sqrt{2}^{\frac{p}{(p-1)}}|x_2 - \frac{1}{2}|^{\frac{p}{p-1}}\right).}$$

Again, one can improve things by addition of a corrector :

$$\boxed{u^\varepsilon(x) \sim u^0(x) + \varepsilon v(x/\varepsilon)}$$

Formally, in the boundary layer.

$$Du^\varepsilon \sim \nu\left(Du^0|_{x_2=0} + Dv(y)\right), \quad y = x/\varepsilon$$

We denote $A := D(u^0)|_{y_2=0^+} = \frac{1}{2}\begin{pmatrix} 0 & U'(0) \\ U'(0) & 0 \end{pmatrix}.$

Boundary layer system of the type:

$$\begin{cases} -\mathrm{div}\, S(A + Dv) + \nabla q = 0 & \text{in } \Omega_{bl}, \\ \mathrm{div}\, v = 0 & \text{in } \Omega_{bl}, \\ v|_{\Gamma_{bl}} = v_0. \end{cases} \tag{BL}$$

Again, one can show exponential convergence of v to $v_\infty = (V, 0)$.

One can show that the best homogenized condition is of the form

$$u_2 = 0, \quad u_1 = \varepsilon \mathcal{F}\left(\partial_2 u_1|_{y_2=0}\right)$$

$\mathcal{F}$ is a nonlinear functional connected to the boundary layer pb.

Second example: Rotating fluids

1. Rotating NS equations

Context: A fluid between two planes, in rotation.

In the rotating frame, two pseudo-forces

- *The centrifugal force*: $\boxed{\rho\,\omega^2\,\nabla(x_1^2 + x_2^2)}$

  Transparent in incompressible models !

- *The Coriolis force* : $\boxed{e \times u}$ with $e = e_3$.

Rotating NS:

$$\boxed{\begin{cases} (\partial_t u + u \cdot \nabla u) + \Omega e \times u + \dfrac{\nabla p}{\rho} - \nu \Delta u = 0, \\ \operatorname{div} u = 0, \\ u|_{x_3=0,L} = 0. \end{cases}}$$

Dimensional analysis:

$$x = Lx', \quad u = Uu', \quad t = \frac{L}{U}t', \quad p = \rho U^2 p'$$

Dropping the primes, one finds

$$\boxed{\begin{cases} \mathrm{Ro}\,(\partial_t u + u\cdot\nabla u) + \nabla p + e\times u - E\Delta u = 0, \\ \mathrm{div}\, u = 0, \\ u|_{x_3=0,1} = 0. \end{cases}}$$

$\mathrm{Ro} := \frac{U}{\Omega L}$: Rossby number. $E := \frac{\nu}{\Omega L^2}$: Ekman number.

Remark: possible variations, inspired by geophysics.

Variation 1: Top plane corresponds to ocean surface.

Rigid lid approximation, forcing by the wind :

$D(u)n \times n|_{x_3=0} = f, \quad u\cdot n|_{x_3=0} = 0.$

Variation 2: Anisotropic eddy viscosities : ν_h, ν_3. Often : $\nu_3 \ll \nu_h$ (the Ekman number is then based on ν_3).

Crucial point: Ro and E are small parameters:

- large scale oceanic or atmospheric motions ($L = 10^5 \mathrm{m}$)

$$\boxed{\mathrm{Ro} \sim 10^{-2} - 10^{-1}, \quad E \sim 10^{-2}}$$

- Earth's core:

$$\boxed{\mathrm{Ro} \sim 10^{-7}, \quad E \sim 10^{-15}}$$

In what follows, for simplicity: $\mathrm{Ro} = \varepsilon$, $E = \varepsilon^2$, $\quad \varepsilon \ll 1$.

$$\boxed{\begin{cases} \partial_t u + u \cdot \nabla u + \dfrac{e \times u}{\varepsilon} + \dfrac{\nabla p}{\varepsilon} - \varepsilon \Delta u = 0, \\ \operatorname{div} u = 0, \\ u|_{x_3=0,1} = 0. \end{cases}} \qquad \text{(NSC)}$$

Remark: Standard results on Navier-Stokes transpose to this case. The Coriolis term disappears from energy estimates.

Weak convergence of u^ε in $L^\infty(L^2)$. Description of the limit u^0 ?

Mathematical interest:

- Penalized operator: $\varepsilon^{-1}\mathbb{P}(e \times \cdot)$. Skew-symmetric over L^2_σ.

  Generates high frequency waves.

  Analogy with weakly compressible flows (acoustic waves).

- Vanishing diffusion : $-\varepsilon\Delta u$. In domains with boundaries, antagonism between the Dirichlet condition and the behaviour of the formal limit u (that is in the kernel of the Coriolis operator).

  $\rightarrow$ *Ekman boundary layers.*

2. The Ekman layer

Question: Asymptotic behaviour of u^ε ?

Weak compactness : u^0 satisfies the *geostrophic balance*

$$\boxed{e \times u^0 + \nabla p^0 = 0, \quad \operatorname{div} u^0 = 0}$$

and the boundary condition $\boxed{u^0 \cdot n = 0}$.

Applying the curl to the first equation yields: $\partial_3 u^0 = 0$.

Finally : $\boxed{u^0 = (u_h(t, x_h, 0) = (u_1(t, x_h), u_2(t, x_h), 0)}$

Incompatible with the Dirichlet condition: Gradients of u^ε must explode near the boundary as $\varepsilon \to 0$, in a boundary layer.

Formal asymptotic expansion:

$$\boxed{u^\varepsilon(t,x) \approx u^0(t,x) + u^0_-\left(t, x_h, \frac{x_3}{\varepsilon}\right) + u^0_+\left(t, x_h, \frac{x_3}{\varepsilon}\right)}$$

$u_\pm = u^0_\pm(t, x_1, x_2, z)$: *boundary layer correctors.* '

Defined for $z \in \mathbb{R}_+$. One expects $u^0_\pm \xrightarrow[z\to+\infty]{} 0$

From the divergence equation : $\boxed{u^0_{\pm,3} = 0}$

<u>Equation for the horizontal part</u>: taking $v = (u_{-,1}, u_{-,2})$,

$$\boxed{v^\perp - \partial_z^2 v = 0}$$

Simple ODE ! t, x_1, x_2 are just parameters.

Boundary condition :

$v_1(t, x_h, 0) = -u^0_1(t, x_h),\ v_2(t, x_h, 0) = -u^0_2(t, x_h)$.

The solution is the famous *Ekman spiral*:

$$\boxed{(v_1 + iv_2)(z) = -(u^0_1 + iu^0_2) \exp\left(-\frac{1 + i)z}{\sqrt{2}\varepsilon}\right)}$$

Question: Dynamics away from the boundary ? Equation on u^0 ?

Go on with the expansion :

$$u^\varepsilon \sim u^0 + u^0_- + u^0_+ \; + \; \varepsilon(u^1 + u^1_- + u^1_+)$$

From the divergence-free condition: $\partial_1 u^0_{\pm,1} + \partial_2 u^0_{\pm,2} \pm \partial_z u^1_{\pm,3} = 0$.

Allows to compute explicitly $u^1_{\pm,3}$.

Back to the interior:

$$\boxed{\partial_t u^0_h + u^0_h \cdot \nabla_h u^0_h + u_1^\perp + \nabla_h p^1 = 0}.$$

Introducing $\omega^0 = \partial_1 u^0_2 - \partial_2 u^0_1$:

$$\partial_t \omega^0 + u^0_h \cdot \nabla_h \omega_0 - \partial_3 u^1_3 = 0.$$

Integrate between $x_3 = 0$ and $x_3 = 1$.

$$\partial_t \omega^0 + u^0_h \cdot \nabla_h \omega_0 + u^1_{+,3}|_{z=0} - u^1_{-,3}|_{z=0} = 0.$$

A little computation provides:

$$\boxed{\partial_t \omega^0 + u_h^0 \cdot \omega^0 + \sqrt{2}\omega^0 = 0}$$

Damped Euler, due to *Ekman pumping*.

Question: Rigorous justification of this limit ?

Need to compare the exact solution to the boundary layer approximation

$$u_a^\varepsilon \; = \; u^0(x) + u_-^0\left(x_1, x_2, \frac{x_3}{\varepsilon}\right) + u_+^0\left(x_1, x_2, \frac{x_3}{\varepsilon}\right) + \ldots$$

Hope:

$$\| u^\varepsilon|_{t=0} - u_a^\varepsilon|_{t=0} \|_{L^2} \to 0 \quad \Rightarrow \quad \sup_{t\in[0,T]} \| u^\varepsilon - u_a^\varepsilon \|_{L^2} \to 0.$$

Remark: We consider *well-prepared initial data*

Perturbation $v^\varepsilon = u^\varepsilon - u_a^\varepsilon$ satisfies

$$\partial_t v^\varepsilon + (u_a^\varepsilon + v^\varepsilon)\cdot\nabla v^\varepsilon + \frac{\nabla q^\varepsilon + e\times v^\varepsilon}{\varepsilon} + v^\varepsilon\cdot\nabla u_a^\varepsilon - \varepsilon\Delta v^\varepsilon = 0$$

Energy estimate:

$$\frac{1}{2}\|v^\varepsilon(t)\|_{L^2}^2 + \varepsilon \int_0^t \|\nabla v^\varepsilon\|_{L^2(\Omega)}^2 \leq \frac{1}{2}\|v^\varepsilon(0)\|_{L^2}^2 + \int_0^t \int |v^\varepsilon|^2 |\nabla u_a^\varepsilon|$$

<u>Pb</u>: $|\nabla u_a^\varepsilon| \approx \frac{1}{\varepsilon}|\partial_z u_-^0\left(\frac{x_3}{\varepsilon}\right)|$

(neglecting the upper boundary layer). Naive control gives

$$\frac{1}{2}\|v^\varepsilon(t)\|_{L^2}^2 \leq \frac{1}{2}\|v^\varepsilon(0)\|_{L^2}^2 e^{\frac{C}{\varepsilon}t}$$

Better idea:

$$\begin{aligned}\int |v^\varepsilon|^2 |\nabla u_a^\varepsilon| &\leq \varepsilon \int \frac{|v^\varepsilon|^2}{(x_3)^2}\frac{(x_3)^2}{\varepsilon^2}|\partial_z u_-^0\left(\frac{x_3}{\varepsilon}\right)| \\ &\leq \varepsilon \sup_{z\in\mathbb{R}^+} |z^2 \partial_z u_-^0(z)| \int \frac{|v^\varepsilon|^2}{(x_3)^2} \\ &\leq C\varepsilon \sup_{z\in\mathbb{R}^+} |z^2 \partial_z u_-^0(z)| \int |\partial_3 v^\varepsilon|^2 \text{ (Hardy inequality)}\end{aligned}$$

Controlled by the diffusion term if $\sup_{z\in\mathbb{R}^+}|z^2\partial_z u_-^0(z)|$ small enough...

Back to units: the stability estimate is obtained if

$$\boxed{R \ := \ \frac{U\|\sup_{t,x_1,x_2} u^0\|_{L^\infty} L\varepsilon}{\nu}}$$

is small enough.

It is a Reynolds number based on the boundary layer length.

<u>Idea</u>: the keypoint is the stability of the normalized Ekman spiral: $u_- = (v_1(z), v_2(z), 0)$ with

$$v_1 + iv_2(z) = e^{-\frac{1+i}{\sqrt{2}}z}$$

seen as a solution of

$$\boxed{\begin{cases} \partial_t u_- + u_- \cdot \nabla u_- + e \times u_+ \nabla p_- - \dfrac{1}{R}\Delta U = 0 \\ \operatorname{div} U = 0. \end{cases}}$$

The threshold is the critical Reynolds number R_c of spectral stability for the linearized equation :

$$\boxed{\partial_t u_- + V \cdot \nabla u_- + u_- \cdot \nabla u_- \nabla p_- - \frac{1}{R}\Delta u_- = 0}$$

- Convergence of u^ε to u^0 if $R \le R_c$: [Rousset'2005].
- Non-convergence if $R > R_c$: [Desjardins-Grenier'2000].

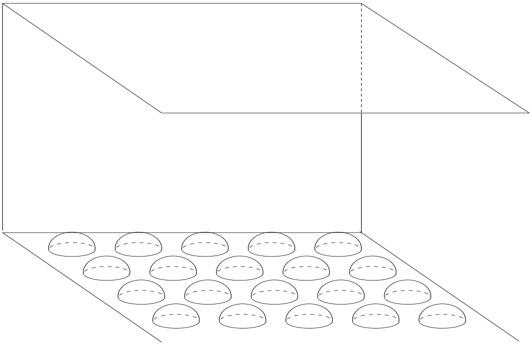
Back to the main topic of the talks...

Question : How is the Ekman layer affected by roughness ?

3. Couche d'Ekman rugueuse

$$\boxed{\Omega^\eta := \left\{x,\ x_h = (x_1, x_2) \in \mathbb{R}^2, \quad 1 > x_3 > \eta\gamma(x_h/\eta)\right\}}$$

$\gamma = \gamma(y_h)$ is Lipschitz, bounded and *periodic*: $y_h = (y_1, y_2) \in \mathbb{T}^2$.



We choose the scaling $\boxed{\mathrm{Ro} \approx E^{1/2} \approx \eta}$ and call ε this common parameter. This choice of scaling is the richest.

$$\boxed{\begin{cases} \partial_t u + u \cdot \nabla u + \dfrac{e \times u}{\varepsilon} + \dfrac{\nabla p}{\varepsilon} - \varepsilon \Delta u = 0, \\ \operatorname{div} u = 0, \\ u|_{\partial \Omega^\varepsilon} = 0. \end{cases}} \tag{NSC}$$

Theorem: *Let $T > 0$. For well-prepared and small enough initial data u_0^ε, u^ε converges in $L^\infty(0, T; L^2)$ to $u^0(t, x) = (u_h(t, x_h), 0)$ satisfying*

$$\boxed{\partial_t u_h + u_h \cdot \nabla u_h + \nabla p + \beta(u_h) = 0, \quad \operatorname{div} u_h = 0 \quad \text{in } \mathbb{R}^2}$$

where $\beta : B(0, \delta) \subset \mathbb{R}^2 \mapsto \mathbb{R}^2$ is defined for small $\delta > 0$ and dissipative: $\boxed{\beta(U) \cdot U > 0 \text{ for all } U \in \mathbb{R}^2 \setminus \{0\}}$.

Remark: Without roughness: $\beta(v) = \sqrt{2}v$. In such case, possible global results in time.

Ideas of the proof

New asymptotic expansion. Neglecting the upper layer:

$$\boxed{u^\varepsilon(x) = u^0(t, x_h) + v\left(t, x_h, \frac{x}{\varepsilon}\right)}$$

with $v = v(t, x_h, y) = v(t, x_h, y_1, y_2, y_3)$: *boundary layer corrector.*

Boundary layer system, in $\Omega_{bl} = \{y,\ y_3 > \gamma(y_1, y_2)\}$:

$$\left\{\begin{aligned} (v + \varphi) \cdot \nabla v + \nabla p + e \times v - \mu \Delta v = 0 \quad & \text{in } \Omega_{bl} \\ \operatorname{div} v = 0 \quad & \text{in } \Omega_{bl} \\ v|_{\partial \Omega_{bl}} = -\varphi. & \end{aligned}\right. \qquad \text{(BL2)}$$

with $\varphi = u^0(t, x_h) \in \mathbb{R}^2 \times \{0\}$.

Remarks:

- $\nabla = \nabla_y$, $\Delta = \Delta_y$: PDE in variable y, parametrized by t, x_h.
- Linear ODE replaced by nonlinear PDE !

Proceeding with the same methodology as in the flat case, we find:

$$\boxed{\beta(U) = \int_{\Omega_{bl}} e \times v_U}$$

where v_U is the solution of (BL2) associated to $\varphi = (U, 0)$.

One can show : $\beta(U) \cdot U = \int_{\Omega_{bl}} |\nabla v_U|^2 > 0$ for $U \neq 0$.

The keypoint is the analysis of (BL2).

Theorem: *For $|\varphi|$ small enough, there exists a unique v such that*

$$\int_{\mathbb{T}^2} \int_{\gamma(y_1,y_2)} |\nabla v(y)|^2 dy_3 dy_1 dy_2 < +\infty$$

v and its derivatives decay exponentially fast as $y_3 \to \infty$.

Remark: Periodicity simplifies greatly the analysis.

- Well-posedness: variational formulation, in a space of periodic functions in y_h.
- Exponential decrease: compactness in y_h.

  Analogue to the case of a channel (although vertical).
  Poincaré for functions with zero horizontal average.

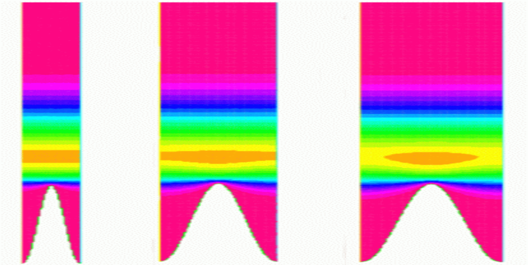
  Estimates of Ladyzenskaya-Solonnikov can be adapted :
  Saint-Venant estimates

Roughness effect on dissipation

Theoretical and numerical study with E. Dormy (linearized): for some configurations, roughness may decrease the dissipation:

Example : "Riblets" (one invariant direction)

- The imposed flow should be along the invariant direction
- The wavelength of the roughness should be
  - neither too long (Ekman layer near an inclined plane).
  - neither too short (fluid is kicked out of the roughness).



Question : Quid about non-periodic roughness ?

Much more difficult : methods of (BL) do not apply to (BL2).

Work in progress with A.L. Dalibard. Variation of (BL2) :

$$\boxed{\begin{cases} v \cdot \nabla v + \nabla p + e \times v - \Delta v = 0 \quad \text{in } \Omega_{bl} \\ \operatorname{div} v = 0 \quad \text{in } \Omega_{bl} \\ v|_{\partial \Omega_{bl}} = \varphi \in \mathbb{R}^2 \setminus \{0\}. \end{cases}} \qquad \text{(BL3)}$$

"Conjecture" : *For* $|\varphi|$ *small enough, system* (BL3) *has a unique solution* $v \in H^1_{loc}(\overline{\Omega_{bl}})$ *with*

$$|v(y)| \leq C(1+y_3)^{-1/3}, \quad \forall y \in \Omega_{bl}.$$

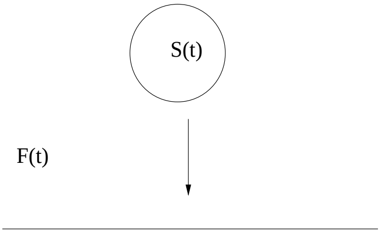
Remark: Loss of exponential decrease. CV to zero persists.

Remark: The tentative proof uses results from the linearized analysis [Dalibard et Prange'2014].

4. Drag computation for rough solids close to contact

Start:

A ball, in a viscous fluid, falling above a wall under the action of gravity.



Fluid and solid at time t : $F(t), S(t)$.

Question : Does the ball touch the wall ?

Archimedes ($\sim$ 265 B.C.): *If* $\rho_S > \rho_F$, *collision.*

Relies on the hydrostatic approximation :

$$\text{Stress tensor :} \qquad \Sigma := (-p_{atm} - \rho_F\, g\, z)\, I_3.$$

Force on the ball :

$$f = -\rho_S g e_z \,|S(t)| + \int_{\partial S(t)} \Sigma n = (\rho_F - \rho_S)\; g\, |S(t)|\, e_z.$$

Pb : Molecular pressure and viscosity are neglected.

Refined model :

- Stokes or Navier-Stokes for the liquid.
- Classical laws of mechanics for the solid.
- *The stress tensor at the solid surfaces includes the newtonian tensor of the fluid.*

Surprise : *In this framework, there is no collision between the sphere and the wall !!*

Shown by [Brenner et al, 1963], [Cooley et al, 1969] for steady Stokes flow.

Shown by [Hillairet'2005] for unsteady Navier-Stokes flow.

Question : What is the flaw of the Navier-Stokes model ? Why is the drag overestimated ?

Refs : [Davis et al, 1986], [Barnocky et al, 1989], [Smart et al, 1989], [Davis et al, 2003].

Idea : Nothing is as smooth as a sphere. *The irregularity of the solid surface can change the solids' dynamics.*

Aim: To obtain an approximate expression for the drag, for various models of roughness.

Pb: The original method of [Brenner et al, 1963] seems hard to transpose.

One needs a method of drag computation not restricted to simple geometries.

Joint work with Matthieu Hillairet.

The method extends partially to Navier-Stokes flows, but for the talk: Stokes flow.

3. "Approximate variational method" for drag computation

One rough solid above a rough wall.

$S(t)$: rough sphere. P: rough plane. Fluid: $F(t)$.

We denote $h(t) := \text{dist}\,(S(t), P)$.

Restriction: the solid translates along a vertical axis.

Remarks:

- One needs good symmetry properties for the solid and the wall. They will be satisfied in our models.
- The geometry of the domain in characterized by h:

$$S(t) = S_{h(t)} = h(t)\,e_z + S, \quad F(t) = F_{h(t)},$$

$S_h = h\,e_z + S$, F_h: domains frozen at distance h.

Equations:

- Stokes equations in the fluid: $x \in F(t), t > 0$:

$$-\Delta u + \nabla p = 0, \quad \operatorname{div} u = 0.$$

- Classical mechanics for the solid:

$$\ddot{h}(t) = \int_{\partial S(t)} (2D(u)n - pn)\, d\sigma \cdot e_z$$

n : outward normal, $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$.

Boundary conditions: will have the following general form:

- No penetration: $u \cdot n|_P = 0, \quad \left(u - \dot{h}(t)\, e_z\right) \cdot n|_{\partial S(t)} = 0$.

- Tangential stress

$$\begin{cases} u \times n|_P = -2\, \beta_P\, D(u)n \times n|_P, \\ (u - \dot{h}(t)\, e_z) \times n|_{\partial S(t)} = -2\, \beta_S\, D(u)n \times n|_{\partial S(t)}. \end{cases}$$

$\beta_S, \beta_P \geq 0$: slip lengths.

If $= 0$: no-slip (Dirichlet). If > 0: slip (Navier).

<u>Crucial remark</u>: This system turns into an ODE

$$\ddot{h}(t) \;=\; -\dot{h}(t)\, f_{h(t)}. \tag{ED}$$

with drag

$$\boxed{f_h \;=\; -\int_{\partial S_h} (2D(u_h)n - p_h n)\; d\sigma \cdot e_z}$$

where (u_h, p_h) solution of

$$\begin{cases} -\Delta u_h + \nabla p_h \;=\; 0, \quad \operatorname{div} u_h = 0, \\ u_h \cdot n|_P = 0, \quad (u_h - \; e_z) \cdot n|_{\partial S_h} = 0, \\ u_h \times n|_P \;=\; -2\;\beta_P\; D(u_h)n \times n|_P \\ (u_h - \; e_z) \times n|_{\partial S_h} \;=\; -2\;\beta_S\; D(u_h)n \times n|_{\partial S_h} \end{cases} \tag{S}$$

Remark: One can forget about the dynamics.

Goal: Study of f_h, h small, for various models of roughness.

Model 1: Non-smooth surface.

Cylindrical coordinates : (r, θ, z).

- $P : \{z = 0\}$
- S : ball of radius 1, perturbed near the south pole by a $C^{1,\alpha}$ "tip", $0 < \alpha < 1$. Locally, for $r < r_0$:

$$\boxed{z = 1 - \sqrt{1 - r^2} \, + \, \varepsilon r^{1+\alpha}}$$

- $\beta_P = \beta_S = 0$.

Remark: Despite this irregularity, $(\nabla u_h, p_h)$ is smooth enough ($W^{s,\tau}$ with $s > 1/\tau$) to define f_h.

Model 2: Wall law of Navier type.

- $P : \{z = 0\}$.
- S : ball of radius 1.
- β_P or $\beta_S > 0$.

Model 3: Oscillations of small amplitude and wavelength.

- $P : \{z = \varepsilon\gamma\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\}$,
  with γ periodic, smooth, ≤ 0, $\gamma(0,0) = 0$.
- S : ball of radius 1.
- $\beta_P = \beta_S = 0$.

Remark: The study is limited to the case $\varepsilon \ll h$.

Remark: Limit case : $\varepsilon \to 0,\ \beta_S, \beta_P \to 0$:

One recovers the well-known case of a sphere and a plane.
Cooley-O'Neil, Cox-Brenner:

$$\boxed{f_h \sim \frac{6\pi}{h}, \quad h \to 0.}$$

(which implies no-collision).

The study of roughness effects requires an approach that is not restricted to simple geometries.

Proposition (Expression of the drag for model 1):

Let $\beta := \varepsilon h^{\frac{\alpha-1}{2}}$.

- In the regime $h \to 0$, $\beta \to 0$:

$$\boxed{f_h \sim \frac{6\pi}{h}(1 + c\,\beta)} \quad c = c(\alpha) \text{ explicit.}$$

- In the regime $h \to 0$, $\beta \to \infty$ (and $\varepsilon = O(1)$):
  - If $\alpha > \frac{1}{3}$,

$$\boxed{f_h \sim c\,\varepsilon^{\frac{-4}{1+\alpha}}\, h^{-\frac{3\alpha-1}{\alpha+1}}} \quad c = c(\alpha) \text{ explicit.}$$

  - If $\alpha = \frac{1}{3}$,

$$\boxed{f_h \sim c\,\varepsilon^{-3}\,|\ln h|} \quad c \text{ explicit.}$$

  - If $\alpha < \frac{1}{3}$,

$$\boxed{f_h = c\,\varepsilon^{\frac{-2}{1-\alpha}} + O(|\ln \varepsilon|)} \quad c = c(\alpha) \text{ explicit}$$

Remarks:

- Collisions are allowed by the model for all $\alpha < 1$. Not allowed for $C^{1,1}$ boundaries.
- The more the boundary is irregular, the less the drag is.
- One recovers the classical result as $\varepsilon = 0$ (with a much simpler proof).

Proposition (Expression of the drag for model 2):

- In the regime $h \to 0$, $\beta_S, \beta_P = O(1)$, with h/β_S or h/β_P uniformly lower bounded, one has

$$\boxed{\frac{c}{h} \le f_h \le \frac{C}{h}} \quad c, C > 0.$$

- In the regime $h \to 0$, $\beta_S, \beta_P = O(1)$, with $h/\beta_S \to 0$ and $h/\beta_P \to 0$, one has

$$\boxed{f_h = 2\pi\left(\frac{1}{\beta_S} + \frac{1}{\beta_P}\right)|\ln h| + O\left(\frac{1}{\beta_S} + \frac{1}{\beta_P}\right)}$$

Remark:

- This roughness model also allows for collision, if β_P *and* $\beta_S > 0$.
- Agrees with formal calculations of Hocking (1973)

Proposition (Expression of the drag for model 3):

In the regime $\varepsilon \ll h \ll 1$:

$$\boxed{\frac{6\pi}{h + c\,\varepsilon} + O(|\ln(h+\varepsilon)|) \le f_h \le \frac{6\pi}{h} + O(|\ln h|)}$$

Remark: With homogenization techniques, one has

$$\boxed{f_h \sim \frac{6\pi}{h + \alpha\varepsilon}}$$

(if $\varepsilon/h \to 0$ fast enough.)

α explicit, associated to some boundary layer problem.

3. Sketch of proof

Step 1: Variational characterization of the drag

$$f_h \ = \ \min_{u \in \mathcal{A}_h} \mathcal{E}_h(u) \ = \ \mathcal{E}_h(u_h).$$

for a good energy functional $\mathcal{E}_h$ and a good admissible set $\mathcal{A}_h$.

<u>Dirichlet case</u> (Models 1 and 3): $\quad \mathcal{E}_h(u) \ := \ \int_{F_h} |\nabla u|^2, \quad$ and

$$\mathcal{A}_h \ := \ \left\{ u \in H^1_{loc}(F_h), \quad \operatorname{div} u = 0, \quad u|_P = 0, \quad u|_{\partial S_h} = e_z \right\}.$$

<u>Navier case</u> (Model 2):

$$\mathcal{E}_h(u) := \int_{F_h} |\nabla u|^2 + \frac{1}{\beta_P} \int_P |u \times n|^2 + \left(\frac{1}{\beta_S} + 1 \right) \int_{\partial S_h} |(u - e_z) \times n|^2,$$

$$\mathcal{A}_h \ := \ \left\{ u \in H^1_{loc}(F_h), \ \operatorname{div} u = 0, \ u \cdot n|_P = (u - e_z) \cdot n|_{\partial S_h} = 0 \right\}.$$

Step 2: Approximate computation of f_h, *via* some relaxed minimization problem.

Rough idea: To find $\tilde{\mathcal{E}}_h \leq \mathcal{E}_h$, and $\tilde{\mathcal{A}}_h \supset \mathcal{A}_h$, such that:

1. $\min_{u \in \tilde{\mathcal{A}}_h} \tilde{\mathcal{E}}_h(u)$ and the associate minimizer can be computed easily.
2. The minimizer $\tilde{u}_h$ belongs to $\mathcal{A}_h$.

It follows that:

$$\tilde{\mathcal{E}}_h(\tilde{u}_h) \;\leq\; f_h \;\leq\; \mathcal{E}_h(\tilde{u}_h)$$

If the relaxed pb is close enough to the original one, it yields a good approximation of the drag.

Remark: this rough idea requires a few adaptations: modification of the minimizer $\tilde{u}_h$ to have it belong to $\mathcal{A}_h$, . . .

Remark: The difficulty lies in the choice of the good relaxed problem.

Example: Model 1 ($C^{1,\alpha}$ tip).

Idea: Simplification due to axisymmetry. The minimizer $u = u_h$ reads

$$u \; = \; -\partial_z \phi(r,z) e_r \; + \; \frac{1}{r} \partial_r (r\phi) e_z. \tag{R}$$

with $\phi = -\int_0^z u_r$. One restricts to fields in $\mathcal{A}_h$ of the type (R).

Boundary conditions on ϕ:

- Wall:

$$\partial_z \phi(r,0) = 0, \quad \phi(r,0) = 0, \tag{cl1}$$

- Near the south pole:

$$\partial_z \phi(r, h + \gamma_\varepsilon(r)) = 0, \quad \phi(r, h + \gamma_\varepsilon(r)) = \frac{r}{2}, \quad r < r_0 \tag{cl2}$$

where $\gamma_\varepsilon(r) = 1 - \sqrt{1 - r^2} + \varepsilon r^{1+\alpha}$.

$$\mathcal{E}_h(u) \;=\; \int_{F_h} |\partial_z^2 \phi|^2 \;+\; \int_{F_h} |\partial_{rz}^2 \phi|^2 \;+\; \dots$$

Idea: The first term is the leading one. Only the zone near $r = 0$ matters.

Relaxed problem:

$$\tilde{\mathcal{A}}_h = \Big\{ u \in H^1_{loc}(F_h),\ \text{satisfying (R)-(cl1)-(cl2)} \Big\},$$

$$\tilde{\mathcal{E}}_h(u) \;=\; \int_0^{r_0} \int_0^{\gamma_\varepsilon(r)} |\partial_z^2 \phi|^2 \, dz \, dr$$

1D minimization problems in z, parametrized by r. Minimizer:

$$\tilde{\phi}_h(r,z) = \frac{r}{2} \Phi\Big(\frac{z}{h + \gamma_\varepsilon(r)}\Big), \quad \Phi(t) = t^2(3 - 2t).$$

The minimum for the relaxed problem (lower bound for f_h) is

$$\tilde{f}_h \;=\; 12\pi \int_0^1 \frac{r^3 dr}{(h+\gamma_\varepsilon(r))^3}\, dr$$
$$\;=\; 12\pi \int_0^1 \frac{r^3 dr}{(h+\frac{r^2}{2}+\varepsilon r^{1+\alpha})^3}\, dr \;+\; \ldots \qquad = \mathcal{I}(\beta) \;+\; \ldots$$

with $\beta \;:=\; \varepsilon\, h^{\frac{\alpha-1}{2}}$, and

$$\mathcal{I}(\beta) \;:=\; \int_0^{+\infty} \frac{s^3 dr}{(1+\frac{s^2}{2}+\beta s^{1+\alpha})^3}.$$

Integral with a parameter, the asymptotics of which can be computed in all regimes.

Similar drag computations are available for the other models.