Symmetric structure of Green–Naghdi equations and global existence for small data of the viscous system

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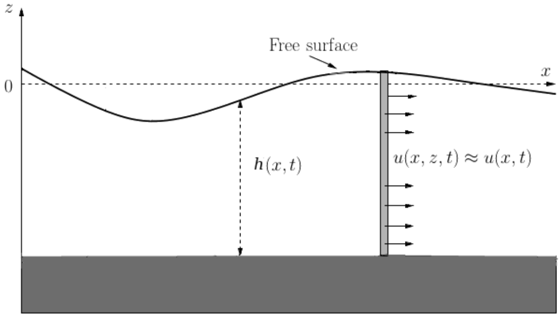
Ecole EGRIN 2015
4^{th} of June

Introduction of the equations

We consider the Green-Naghdi equations,

$$\begin{cases} \partial_t h + \partial_x hu = 0, \\ \partial_t hu + \partial_x hu^2 + \partial_x (gh^2/2 + \alpha h^2 \ddot{h}) = 0. \end{cases} \quad (1)$$

where h is the water height, u is the horizontal speed and $\alpha > 0$.The material derivative is given by $\dot{()} = \partial_t() + u\partial_x()$.



Plan

1. Symmetric structure of the equations
   - Some reminders about hyperbolic systems
   - Generalization of the notion of symmetry

2. Global existence for small data of the viscous system
   - Results for hyperbolic systems obtained by several authors
   - Global existence for small data of the viscous Green–Naghdi system

Let us consider a one dimensional n-hyperbolic system of conservation law

$$\partial_t U + \partial_x F(U) = 0. \qquad (2)$$

where F is a function defined on an open subset Ω of $\mathbb{R}^n$.

Definition

The system (2) is symmetrizable if there exists a change of variable $U \mapsto V$, a symmetric definite positive matrix $A_0(V)$ and a symmetric matrix $A_1(V)$, such that the system is written under the form

$$A_0(V)\partial_t V + A_1(V)\partial_x V = 0.$$

Some reminders about hyperbolic systems

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Definition

The system (2) admits an entropy in the sense of Lax if there exists a strictly convex function E and a function P defined on Ω such that

$$(\nabla_U F(U))^T \nabla_U E(U) = \nabla_U P(U).$$

Remark

System (2) admits an entropy in the sense of Lax i.e. it is such that

$$(\nabla_U F(U))^T \nabla_U E(U) = \nabla_U P(U),$$

iff the solution U of the system satisfies

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iff the solution U *of the system satisfies*

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Proposition (Godunov 1961)

All entropic hyperbolic systems are symmetrizable under any variable.

This is due to the fact that all entropic hyperbolic systems own a Godunov structure i.e. it is written under

$$\partial_t \left(\nabla_Q E^\star(Q)\right) + \partial_x \left(\nabla_Q \hat{P}(Q)\right) = 0,$$

where

$$E^\star = Q \cdot U - E(U),$$

is the Legendre Transform of E for the change of variable

$$Q = \nabla_U E(U),$$

and

$$\hat{P}(Q) = Q \cdot F(U(Q)) - P(U(Q)).$$

Multidimensional generalization : The symmetric structure and the entropy of the following hyperbolic system,

$$\partial_t U + \sum_{i=1}^{d} \partial_{x_i} F_i(U) = 0, \qquad (3)$$

are respectively defined by

$$A_0(V)\partial_t V + \sum_{i=1}^{d} A_i(V)\partial_{x_i} V = 0,$$

and

$$\nabla_U E(U) \nabla_U F_i(U) = \nabla_U P_i(U) \quad \forall i \in \{1, ..., d\},$$

for a strictly convex function of U and some functions P_i.

Then, a similar proposition holds true.

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   - Results for hyperbolic systems obtained by several authors
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Let us now consider the following general system

$$\partial_t U + \partial_x F(U) = 0, \tag{4}$$

where $U \in C([0,T);\mathcal{A})$ for some $T > 0$ and F is a differentiable application acting on a functional space $\mathcal{A}$ (a subspace of $\mathbb{L}^2(\mathbb{R})$).

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Definition

The system (4) is symmetrizable if there exists a change of variable $U \mapsto V$, a symmetric definite positive operator $A_0(V)$ and a symmetric operator $A_1(V)$, such that the system is written under the form

$$A_0(V)\partial_t V + A_1(V)\partial_x V = 0.$$

Generalization of the symmetrizability

We have for hyperbolic systems

$$\text{Godunov structure} \Leftrightarrow \text{Entropy in the sense of Lax.}$$

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What we do here

$$\text{function } E \mapsto \text{functional } \mathcal{H} = \int_{\mathbb{R}} E$$

$$\text{gradient } \nabla \mapsto \text{variational derivative } \delta.$$

$$\text{Hessienne } \nabla^2 \mapsto \text{second variation } \delta^2.$$

$$\mathcal{H}^\star = \int_{\mathbb{R}} U \cdot \delta_U \mathcal{H} - E(U).$$

$$\text{Godunov structure} \mapsto \text{general Godunov structure.}$$

Generalization of the symmetrizability

Theorem (K. 2014)

Let us assume that there exists a functional $\mathcal{H}(U) = \int_{\mathbb{R}} E(U)$ strictly convex on an open convex subset Ω of $\mathcal{A}$ such that $\delta_U^2 \mathcal{H}(U) D_U F(U)$ is symmetric. Then, (4) owns a general Godunov structure i.e. the system is written under

$$\partial_t(\delta_Q \mathcal{H}^\star(Q)) + \partial_x (\delta_Q \mathcal{R}(Q)) = 0, \tag{5}$$

where

$$Q = \delta_U \mathcal{H}(U),$$

and $\mathcal{R}$ is a functional defined on $\delta\mathcal{H}(\Omega)$.

Generalization of the symmetrizability

Theorem (K. 2014)

Let us assume that (4) owns a general Godunov structure through a strictly convex functional $\mathcal{H}$ of Ω. Then, the system is symmetrizable under any change of unknown $U \mapsto V$ i.e. it is equivalent to

$$A_0(V)\partial_t V + A_1(V)\partial_x V = 0.$$

Moreover, the expressions of the symmetric definite positive operator $A_0(V)$ and the symmetric one $A_1(V)$ are given by

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_V U, \tag{6}$$

and

$$A_1(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_U F(U) D_V U. \tag{7}$$

Generalization of the symmetrizability

Corollary

The three following statements are equivalent :

1. *System (4) owns a general Godunov structure through a strictly convex functional* $\mathcal{H}^\star$.
2. *There exists a strictly convex functional* $\mathcal{H}$ *such that the operator* $\delta_U^2\mathcal{H}(U)D_UF(U)$ *is symmetric.*
3. *System (4) is symmetrizable under any change of unknown* $U \mapsto V$ *of the form* $A_0(V)\partial_t V + A_1(V)\partial_x V = 0$ *where the expressions of* $A_0(V)$ *and* $A_1(V)$ *are given by (6) and (7).*

Generalization of the symmetrizability

Corollary

The three following statements are equivalent :

1. *System (4) owns a general Godunov structure through a strictly convex functional $\mathcal{H}^\star$.*
2. *There exists a strictly convex functional $\mathcal{H}$ such that the operator $\delta^2_U \mathcal{H}(U) D_U F(U)$ is symmetric.*
3. *System (4) is symmetrizable under any change of unknown $U \mapsto V$ of the form $A_0(V)\partial_t V + A_1(V)\partial_x V = 0$ where the expressions of $A_0(V)$ and $A_1(V)$ are given by (6) and (7).*

Remark

The system is symmetrizable only while the solution remains in the domain of convexity of $\mathcal{H}$. We say that the system is locally symmetrizable on a particular solution U_0 if the Hamiltonian $\mathcal{H}$ is strictly convex on a neighborhood of U_0.

- It is based on the decomposition $U = (U_1, U_2)$ of the unknown if the following change of variable is well-defined.
- It is given by the partial variational derivative of the strictly convex functional $\mathcal{H}$ i.e. by

$$U \mapsto V = (V_1, V_2),$$

such that

$$U_1 = V_1 \quad \text{and } V_2 = \delta_{U_2}\mathcal{H}(U).$$

Advantage : The definite positive operator $A_0(V)$ is bloc diagonal.

The link with the conservation law

Question : It is well-known that in the case of hyperbolic systems, the Godunov structure and the existence of an entropy equality are equivalent. Does such an equivalence hold true for the abstract system (4) ?

Proposition (K. 2014)

Let us assume that (4) is a general Godunov system on an open convex subset Ω of $\mathcal{A}$. i.e. there exists a strictly convex functional $\mathcal{H} = \int_{\mathbb{R}} E(U)$ defined on Ω such that, as long as U remains in Ω, the system is written under

$$\partial_t(\delta_Q \mathcal{H}^\star(Q)) + \partial_x (\delta_Q \mathcal{R}(Q)) = 0,$$

for $Q = \delta_U \mathcal{H}(U)$ and a functional $\mathcal{R}(Q) = \int_{\mathbb{R}} R(Q)$ defined on $\delta_U \mathcal{H}(\Omega)$. Then, the solution U satisfies

$$\int_{\mathbb{R}} \partial_t E(U) + \partial_x N(U)\, dx = 0, \text{ with } N(U) = Q(U) \cdot F(U) - R(Q(U)).$$

Contrary to the case of hyperbolic systems, the reciprocal of the proposition is false. This is due to

$$\int_{\mathbb{R}} D_U N(U)\phi = \int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U)\phi \quad \forall \phi \in \mathcal{A} \not\Leftrightarrow$$
$$\int_{\mathbb{R}} D_U N(U)\partial_x U = \int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U)\partial_x U.$$

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Weak symmetry : Moreover, the general Godunov structure of the system does not lead to a conservation law but it leads to a conserved quantity. This is due to the fact that the definition of the generalized symmetry we chose is weak (based on the $\mathbb{L}^2$ scalar product). Therefore, we can call it the weak symmetry.

Multi-dimensional generalization

$$\partial_t U + \sum_{i=1}^{n} \partial_{x_i} F_i(U) = 0. \tag{8}$$

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The following conditions are equivalent

1. There exists a strictly convex functional $\mathcal{H}(U) = \int_{\mathbb{R}} E(U)$ such that $\delta_U^2 \mathcal{H}(U) D_U F_i(U)$ is symmetric for all $i \in \{1, ..., n\}$.
2. System (8) is a general Godunov system. i.e. it is equivalent to

$$\partial_t(\delta_Q \mathcal{H}^\star(Q)) + \sum_{i=1}^{n} \partial_{x_i}(\delta_Q \mathcal{R}_i(Q)) = 0,$$

   for some functionals $\mathcal{R}_i(Q) = \int_{\mathbb{R}} R_i(Q)$ with $i \in \{1, ...n\}$ and a strictly convex functional $\mathcal{H}^\star$.
3. System (8) is symmetrizable under any change of unknown $U \mapsto V$ with

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_V U, \quad A_i(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_U F_i(U) D_V U.$$

Let us now consider again the Green-Naghdi equations,

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2) + \partial_x(gh^2/2 + \alpha h^2 \ddot{h}) = 0. \end{cases} \tag{9}$$

and let us introduce the variable (Li, 2002)

$$m = \mathcal{L}_h(u) = hu - \alpha\partial_x(h^3\partial_x u).$$

Application to the Green–Naghdi equations

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and let us introduce the variable (Li, 2002)

$$m = \mathcal{L}_h(u) = hu - \alpha\partial_x(h^3\partial_x u).$$

The system is equivalent to

$$\partial_t U + \partial_x F(U) = 0,$$

where

$$\begin{cases} U = (h, m), \\ F(U) = \begin{pmatrix} h\mathcal{L}_h^{-1}(m) \\ m\mathcal{L}_h^{-1}(m) - 2\alpha h^3(\partial_x \mathcal{L}_h^{-1}(m))^2 + \frac{g}{2}h^2 - \frac{g}{2}h_e^2 \end{pmatrix}. \end{cases}$$

Application to the Green–Naghdi equations

The Green-Naghdi equations is a particular case of the abstract frame presented before, since F is a twice differentiable application acting on

$$\mathcal{A} = (\mathbb{H}^s(\mathbb{R}) + h_e) \times \mathbb{H}^{s-1}(\mathbb{R})$$

for all integer $s \geq 2$ and all $h_e > 0$.

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Remark

The space $\mathcal{A}$ is also the space of the local well-posedness of the system (Li 2006, Israwi 2011, Lannes 2008).

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After basic computations, we remark that the solution of the system satisfies

$$\partial_t E_{h_e}(U) + \partial_x P_{h_e}(U) = 0,$$

where

$$E_{h_e} = gh(h - h_e)/2 + hu^2/2 + \alpha h^3 (u_x)^2/2,$$

and

$$P_{h_e} = u\left(E_{h_e} + gh^2/2 + \alpha h^2 \ddot{h}\right).$$

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The Green-Naghdi equation is locally symmetrizable on $(h_e, 0)$.

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Proof

- *We consider the energy integral* $\mathcal{H}_{h_e} = \int_{\mathbb{R}} E_{h_e}$, *which is strictly convex on* $(h_e, 0)$.
- *We remark that the system is a general Godunov system of the form*

$$\partial_t \left(\delta_Q \mathcal{H}^{\star}_{h_e}(Q)\right) + \partial_x \left(\delta_Q \mathcal{R}(Q)\right) = 0,$$

where

$$Q = \delta_U \mathcal{H}_{h_e}(U) = (gh - gh_e/2 - u^2/2 - \frac{3}{2}\alpha h^2 (u_x)^2, u),$$

and

$$\mathcal{R}(Q) = \int_{\mathbb{R}} gu \left(\frac{h^2 - h_e^2}{2}\right) - \alpha h^3 u (u_x)^2.$$

Here is the symmetric structure of the system under the physical variable $V = (h, u)$:

$$A_0(V) = \begin{pmatrix} g - 3\alpha h(u_x)^2 & 0 \\ 0 & \mathcal{L}_h \end{pmatrix},$$

and

$$A_1(V) = \begin{pmatrix} gu - 3\alpha hu(u_x)^2 & gh - 3\alpha h^2(u_x)^2 \\ gh - 3\alpha h^2(u_x)^2 & hu + 2\alpha\partial_x(h^3 u_x) - \alpha h^3 u_x \partial_x - \alpha u \partial_x(h^3\partial_x()) \end{pmatrix}.$$

Remark

$A_0(V)$ is diagonal because $V = (h, u)$ is obtained by the partial variational derivative of $\mathcal{H}_{h_e}$ with respect to m i.e

$$(h, u) = (h, \delta_m \mathcal{H}_{h_e}(U)).$$

The symmetric structure of the system under the variable $Q = (gh - gh_e/2 - u^2/2 - \frac{3}{2}\alpha h^2(u_x)^2, u)$ is given by :

$$A_0(Q) =$$

$$\begin{pmatrix} \frac{1}{g-3\alpha h(u_x)^2} & \frac{u+3\alpha h^2 u_x \partial_x}{g-3\alpha h(u_x)^2} \\ \frac{u}{g-3\alpha h(u_x)^2} - 3\alpha\partial_x\left(\frac{h^2 u_x}{g-3\alpha h(u_x)^2}()\right) & \mathcal{L}_h + \left(\frac{u}{g-3\alpha h(u_x)^2}\right)\left(u + 3\alpha h^2(u_x)\partial_x\right) \\ & -3\alpha\partial_x\left(h^2 u_x \frac{u()+3\alpha h^2 u_x \partial_x()}{g-3\alpha h(u_x)^2}\right) \end{pmatrix}$$

$$A_1(Q) =$$

$$\begin{pmatrix} \frac{u}{g-3\alpha h(u_x)^2} & h + \frac{u^2+3\alpha h^2 u u_x \partial_x}{g-3\alpha h(u_x)^2} \\ h + \frac{u^2}{g-3\alpha h(u_x)^2} - 3\alpha\partial_x\left(\frac{h^2 u(u_x)}{g-3\alpha h(u_x)^2}()\right) & 3hu + \frac{u^3+3\alpha h^2 u^2 u_x \partial_x}{g-3\alpha h(u_x)^2} - \alpha\partial_x\left(h^3 u_x()\right) - \alpha u\partial_x\left(h^3\partial_x()\right) \\ & -3\alpha\partial_x\left(\frac{h^2 u^2 u_x + 3\alpha h^4 u(u_x)^2\partial_x()}{g-3\alpha h(u_x)^2}\right) \end{pmatrix}.$$

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- *A 2D generalization is possible.*
- *A generalization to all constant solutions of the form (h_e, u_e) is possible.*

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Hyperbolic-Parabolic Systems by Kawashima–Shizuta

Theorem (Kawashima-Shizuta, 1986)

Let us consider a n-hyperbolic-parabolic system of the form

$$A_0(U)\partial_t U + A_1(U)\partial_x U = B\partial_x^2 U, \tag{10}$$

such that

- ***Symmetrizability :*** *$A_0(U)$ is a symmetric definite positive matrix, $A_1(V)$ is a symmetric matrix.*
- ***Entropy dissipativity :*** *B is a symmetric constant definite positive matrix such that its kernel is invariant under $A_0(U)$.*
- ***Kawashima–Shizuta condition :*** *There exists a real matrix K such that $KA_0(U_e)$ is skew-symmetric and $\frac{1}{2}\left(KA_1(U_e) + A_1(U_e)K^T\right) + B(U_e)$ is definite positive for a constant solution U_e.*

Then, the equilibrium U_e is asymptotically stable for the norm of the space $C([0,\infty);\mathbb{H}^s(\mathbb{R}))$ for all integer $s \geq 2$.

Definition

A particular global solution U_e of an evolution system is called asymptotically stable if there exists a neighborhood of U_e such that for all initial data in this neighborhood, the solution of the system exists for all time and converges to U_e while $t \to \infty$.

Hyperbolic systems with friction

Theorem (Hanouzet–Natalini 2003, Yong 2004)

Let us consider a n system with a friction of the form

$$A_0(U)\partial_t U + A_1(U)\partial_x U = (0, Q(U)) \,, \tag{11}$$

where $U = (U_1, U_2)$ is a n component vector. Let us also consider a constant vector $U_e = (U_e^1, U_e^2)$ such that $Q(U_e) = 0$. We also assume that

- ***Symmetrizability :*** *$A_0(U)$ is a symmetric definite positive matrix, $A_1(V)$ is a symmetric matrix.*
- ***Entropy dissipativity :*** *There exists a definite positive matrix $B(U)$ such that $Q(U) = -B(U)(U_2 - U_e^2)$.*
- ***Kawashima–Shizuta condition :*** *There exists a real matrix K such that $KA_0(U_e)$ is skew-symmetric and*

  $$\frac{1}{2}\left(KA_1(U_e) + A_1(U_e)K^T\right) + \begin{pmatrix} 0 & 0 \\ 0 & B(U_e) \end{pmatrix}$$

  is definite positive.

Then, the equilibrium U_e is asymptotically stable for the norm $C([0,\infty); \mathbb{H}^s(\mathbb{R}))$.

Sketch of the proof for hyperbolic case

The proof is based on two class of estimates :

- The first category is obtained by taking the scalar product of the s^{th} derivative of the system by the s^{th} derivative of the solution using the symmetric structure and the entropy dissipativity.

Sketch of the proof for hyperbolic case

The proof is based on two class of estimates :

- The first category is obtained by taking the scalar product of the s^{th} derivative of the system by the s^{th} derivative of the solution using the symmetric structure and the entropy dissipativity.
- The second category is obtained by acting the operator $K\partial_x^{s-1}$ on the system before taking the scalar product by the s^{th} derivative of the solution.

Combining these two estimate we can find $\delta > 0$ such that for all initial data in the δ-neighborhood of U_e, the solution belongs to the neighborhood far all time.

Is it possible to prove the global existence of the solution of the Green-Naghdi equation with the μ-viscosity by generalizing the techniques already used for hyperbolic systems ?

$$\begin{cases} \partial_t h + \partial_x hu = 0, \\ \partial_t hu + \partial_x hu^2 + \partial_x (gh^2/2 + \alpha h^2 \ddot{h}) = \mu \partial_x (h \partial_x u). \end{cases} \tag{12}$$

The local well-posedness space is for

$$\mathbb{X}^s(\mathbb{R}) = (\mathbb{H}^s(\mathbb{R}) + h_e) \times \mathbb{H}^{s+1}(\mathbb{R}),$$

for some integer $s \geq 2$.

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- $A_0(V)$ and $A_1(V)$ are not matrix but operators of order 2.

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Asymptotic stability of the equilibriums of the GN equations with viscosity

Theorem (K. 2015)

Let us consider the equilibrium $V_e = (h_e, u_e)$ of (12) and $\bar{s} \geq 2$ an integer. Then, there exists $\delta > 0$ such that for all initial data $V_0 = (h_0, u_0) \in B_{\bar{s}}(V_e, \delta)$, the solution V exists for all time and converges asymptotically to V_e.
In other words, every constant solution $V_e = (h_e, u_e)$ of (12) is asymptotically stable.

Notation : $B_{\bar{s}}(V_e, \delta)$ represents the δ-neighborhood for the norm $\mathbb{X}^{\bar{s}}$ of the equilibrium V_e.

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Notation : $B_{\bar{s}}(V_e, \delta)$ represents the δ-neighborhood for the norm $\mathbb{X}^{\bar{s}}$ of the equilibrium V_e. In fact, we just need to prove the theorem for equilibriums of the form

$$V_e = (h_e, 0).$$

This is due to the existence of a special invariance for the system.

Remark (Li, Bagderina, Chupakhin)

The operator $v = t\partial_x + \partial_u$ *is a infinitesimal generator of a symmetry group of* (12) . *That is to say that*

$$V_\beta = (h(x - \beta t, t), u(x - \beta t, t) + \beta)$$

is also a solution of (12) *for all solution* $V = (h, u)$ *and all* $\beta \in \mathbb{R}$.

The key of the proof for the stability of equilibriums is the following proposition :

Proposition (K. 2015)

Assume also that there exists $\bar{T} > 0$ such that the unique local solution V satisfies $V(T) \in B_{\bar{s}}(V_e, \delta)$ for all $0 \leq T < \bar{T}$. Then, we have for all $T \in [0, \bar{T})$,

$$(1 - \Theta_{\{h_e,\alpha\}}(\delta)) \parallel V(T) - V_e \parallel^2_{\mathbb{X}^{\bar{s}}} + C_{\{h_e,\mu\}}(\delta) \int_0^T \parallel u_x \parallel^2_{\mathbb{H}^{\bar{s}}} \leq$$

$$C_{\{h_e,\alpha\}}(\delta) \parallel V(0) - V_e \parallel^2_{\mathbb{X}^{\bar{s}}} + \Theta_{\{h_e,\mu,\alpha\}}(\delta) \int_0^T \parallel u_x \parallel^2_{\mathbb{H}^{\bar{s}}}$$

Notation : Symbol $C_S(\delta)$ stands for a function of δ, defined by the elements of the set S, which converges to a limit strictly different from zero while δ goes to 0.
Symbol $\Theta_S(\delta)$ stands for a function, defined by the elements of the set S, which converges to zero while δ goes to 0.

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$$\mathcal{H}_{h_e}(h(t), u(t)) - \mathcal{H}_{h_e}(h(0), u(0)) = -\mu \int_0^t \int_{\mathbb{R}} h(u_x)^2 \leq 0.$$

This leads us to

$$\| V(T) - V_e \|_{\mathbb{X}^0}^2 + C_{\{h_e\}}(\delta) \int_0^T \| u_x \|_{\mathbb{L}^2}^2 \leq C_{\{h_e\}}(\delta) \| V(0) - V_e \|_{\mathbb{X}^0}^2 .$$

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The higher order estimates are the results of the time and space integral of the scalar product of the s^{th} derivative (for $1 \leq s \leq \bar{s}$) of the symmetric equation and the s^{th} derivative of the solution.

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Estimate on $\int_0^T \| h_x \|_{\mathbb{X}^{\bar{s}-1}}^2$: We introduce the hollow matrix

$$K(V_e) = \begin{pmatrix} 0 & 1 \\ -\frac{h_e}{g} & 0 \end{pmatrix}.$$

We remark that

$$K(V_e)A_1(V_e) + \begin{pmatrix} 0 & 0 \\ 0 & h_e^2 + 1 \end{pmatrix},$$

is symmetric definite positive matrix even though $K(V_e)A_0(V_e)$ is not skew-symmetric. Therefore, we extract a convenient part from $K(V_e)A_0(V_e)$ we know a lower bound of.

Sketch of the proof of Proposition

Therefore, acting $K(V_e)\partial_x^{s-1}$, for $1 \leq s \leq \bar{s}$, on the symmetric equation and taking the scalar product with the s^{th} derivative of the solution, we obtain

$$\int_0^T \| h_x \|^2_{\mathbb{H}^{\bar{s}-1}} \leq C_{\{h_e,\alpha\}}(\delta) \left(\| u(T) \|^2_{\mathbb{H}^{\bar{s}+1}} + \| \partial_x h(T) \|^2_{\mathbb{H}^{\bar{s}-1}}\right)$$
$$+ C_{\{h_e,\mu\}}(\delta) \int_0^T \| u_x \|^2_{\mathbb{H}^{\bar{s}}} + C_{\{h_e,\alpha\}}(\delta) \left(\| u(0) \|^2_{\mathbb{H}^{\bar{s}+1}}\right)$$
$$+ C_{\{h_e,\alpha\}}(\delta) \left(\| \partial_x h(0) \|^2_{\mathbb{H}^{\bar{s}-1}}\right).$$

Remark

The coercivity of $A_0(V_e)$ plays a very important role in the proof. The definite positivity is not sufficient.

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Corollary

(Asymptotic stability of equilibrium solutions) *Let us* $\bar{s} \geq 2$ *be an integer and consider the equilibrium* $V_e = (h_e, 0)$ *of* (12) *. Then, there exists* $\delta > 0$ *such that for all initial data* $V_0 = (h_0, u_0)$ *in* $B_{\bar{s}}(V_e, \delta)$ *, the global solution* $V(x,t)$ *in* $\mathbb{X}^{\bar{s}}(\mathbb{R})$ *of* (12) *converges asymptotically to* V_e*. In other words,* $\lim_{t \to \infty} V(x,t) = V_e$ *for all* $x \in \mathbb{R}$.

Proof of Corollary

We then take the x derivative of the equation, its time integral on $[t_1, t_2]$ and consider the $\mathbb{H}^1 \times \mathbb{L}^2$ norm :

$$\| U_x(t_2) - U_x(t_1) \|_{\mathbb{H}^1\times\mathbb{L}^2} = \| \int_{t_1}^{t_2} \partial_{xx} F(U) + \begin{pmatrix} 0 \\ \mu\partial_x^2(hu_x) \end{pmatrix} \|_{\mathbb{H}^1\times\mathbb{L}^2} .$$

Hence,

$$\| U_x(t_2) - U_x(t_1) \|_{\mathbb{H}^1\times\mathbb{L}^2} \leq$$
$$|t_2 - t_1| \left(\sup_{t_1 \leq t \leq t_2} \| \partial_{xx} F(U) \|_{\mathbb{H}^1\times\mathbb{L}^2} + \mu \sup_{t_1 \leq t \leq t_2} \| \partial_x^2(hu_x) \|_{\mathbb{H}^1\times\mathbb{L}^2} \right) .$$

The proposition together with the continuity of F gives us a $\tilde{C} > 0$ such that we have for all t_1, t_2 positive,

$$\begin{aligned} | \| U_x(t_1) \|_{\mathbb{H}^1\times\mathbb{L}^2} - \| U_x(t_2) \|_{\mathbb{H}^1\times\mathbb{L}^2} | &\leq \| U_x(t_2) - U_x(t_1) \|_{\mathbb{H}^1\times\mathbb{L}^2} \\ &\leq \tilde{C}|t_2 - t_1|. \end{aligned}$$

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Thank you for your attention !