Non-homogeneous incompressible Bingham flows with variable yield stress and application to volcanology.

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Laboratoire de Maths & Laboratoire Magmas et Volcans

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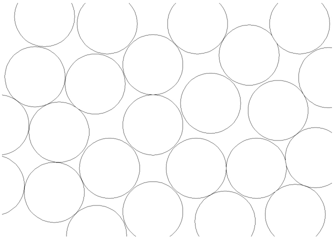
Laboratory experiment

Zoom into the granular column $\rightarrow$

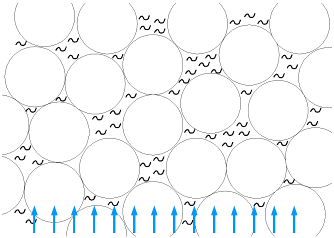


O. Roche, LMV.

Laboratory experiment

Zoom into the granular column →

Fluidisation :
injection of **gas**
through
a pore plate.



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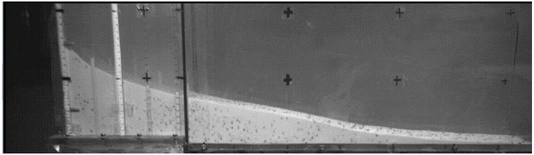

Figure: Non-fluidised → **short** runout distance

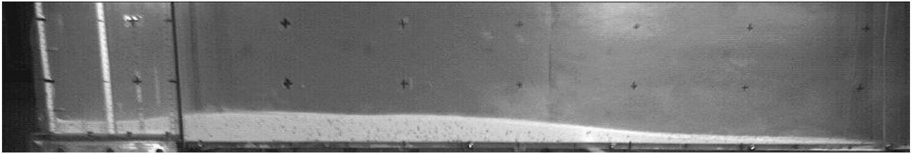

Figure: Fluidised → **long** runout distance

1 Model

Two phases for one fluid

Consider **one mixed fluid** with **variable density**.

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Unknown:

- v: velocity,
- p: total pressure,
- ρ: density.

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Rheology

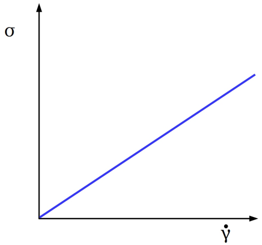
Let precise $\mathbf{S}$.

Rheology

In our case

$$\mathbf{S} = \mu Dv \quad +$$

where $\quad Dv = \dot{\gamma}$ is the strain rate tensor,
μ is the effective viscosity



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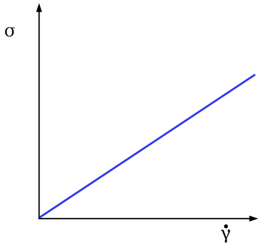
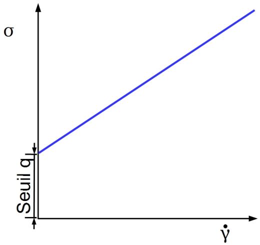
$\Rightarrow$ Bingham fluid with yield stress q





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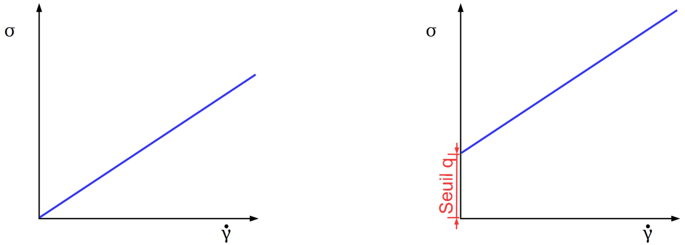

Idea

Let vary the yield stress q as a function of the **interstitial gas pressure**.

Variation of the yield stress

Definition of the yield stress

$$q = \begin{cases} \text{atmospheric pressure:} & \text{Coulomb friction} \\ \text{high pressure:} & \text{fluid} \end{cases}$$

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Equations of the model

Noting p_f the interstitial gas pressure, we obtain:

$$
\begin{cases}
\mathrm{div}(v) = 0 \\
\partial_t \rho + \mathrm{div}(\rho v) = 0 \\
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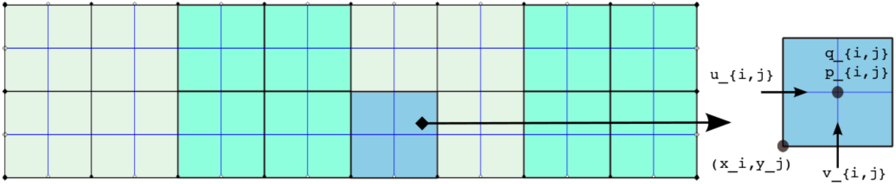
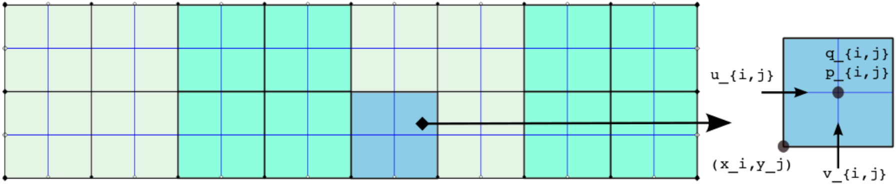
where κ is the diffusion coefficient.

2 Numerical simulation

Dambreak: Mesh



Dambreak: how to treat the density?



$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho = 0$$

numerical method:

RK3 TVD - WENO5 scheme.

Numerical scheme (without density)

$n = 0$ $\qquad$ v^0, Σ^0, p^0 and ${p_f}^0$ given.

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We compute $(\widetilde{v}^{n+1}, \Sigma^{n+1})$ solution of

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We compute (v^{n+1}, p^{n+1}) thanks to the incompressibility constrain

$$\begin{cases} \dfrac{v^{n+1} - \widetilde{v}^{n+1}}{\delta t} + \dfrac{2}{3\rho^{n+1}}\nabla(p^{n+1} - p^n) = \mathbf{0}, \\ \operatorname{div} v^{n+1} = 0, \quad v^{n+1}\cdot\mathbf{n}\Big|_{\partial\Omega} = 0. \end{cases}$$

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Experimental conditions

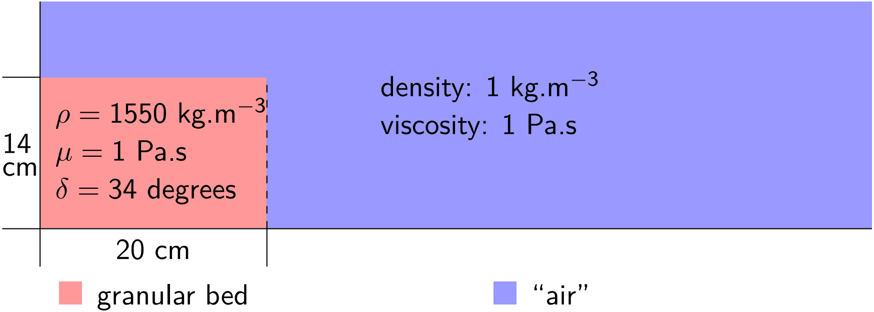

Figure: Experimental setup

Numerical simulation

We compute the collapse with different diffusion coefficient κ.

- With $\kappa = 1$, it happens nothing.
- $\kappa = 0.2$
- $\kappa = 0.1$

Diffusion coefficient = 0.2

Diffusion coefficient = 0.1

3 Perspectives

Perspectives

- To compare the results given by this numerical scheme for the **non-fluidised** case:
  - Lower yield stress,...
  - Lower friction coefficient...
- To compare the results given by this numerical scheme for the **fluidised** case:
  - Adapt the viscosity value.

Thank you!