Modeling of two-phase flow

> Direct steam generation in solar thermal power plants

Pascal Richter
RWTH Aachen University

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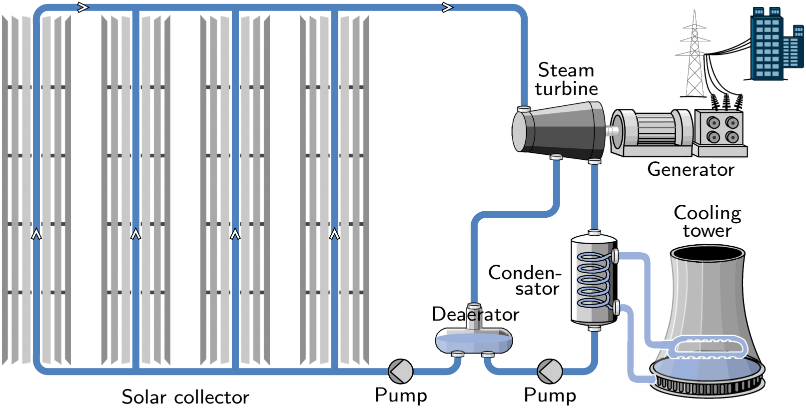
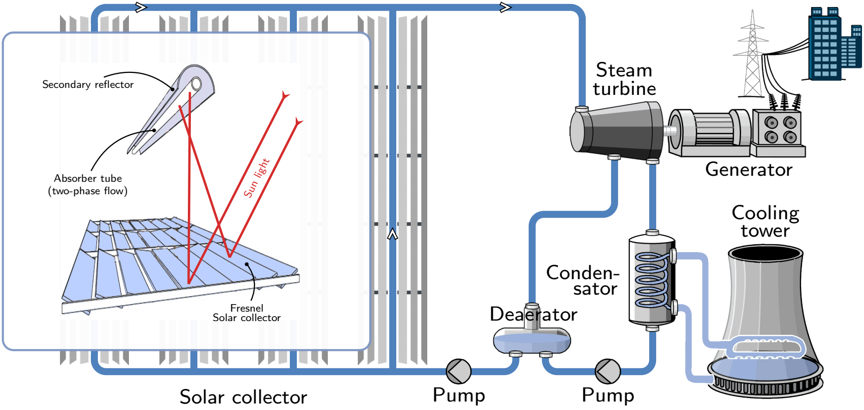
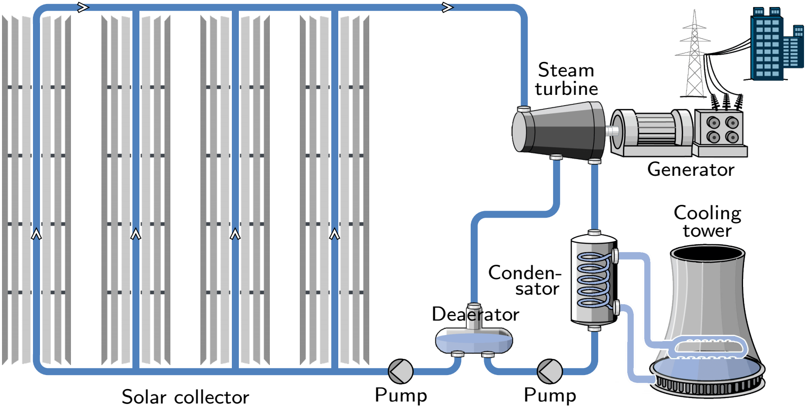
Direct steam generation | Solar thermal power plant



Direct steam generation | Fresnel collector

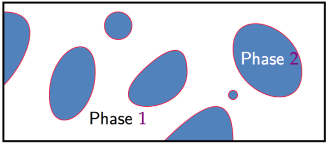


Direct steam generation | Two-phase flow



- Liquid and steam phase in absorber tubes
- Exchange of mass, momentum and energy across the phases
- Interaction of the phases at the wall
- Network coupling

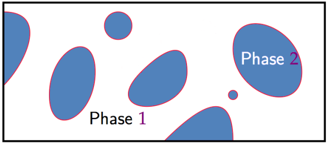
How to model two-phase flow?



Fluid k, that occupies the observed domain, is described with **Navier-Stokes** equations: Continuity, momentum and total energy

Plenty of models in the literature!

How to model two-phase flow?



Model development

- Dimension reduction
- Averaging of the Navier-Stokes equations
- Source terms
- Quantities: Density, Velocity, Energy, Pressure } **separate**, **mixture** or **equal**

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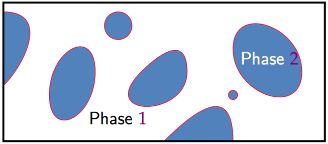

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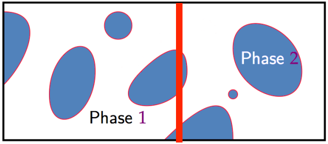

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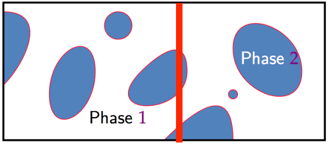

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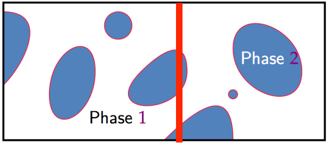

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Two-phase flow model

The system is in non-conservative form

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) + B(\mathbf{u})\,\partial_x \mathbf{u} = \mathbf{s}(\mathbf{u}),$$

$$\mathbf{u} = \begin{pmatrix} \alpha_g \\ \alpha_\ell \rho_\ell \\ \alpha_\ell \rho_\ell v_\ell \\ \alpha_\ell \rho_\ell E_\ell \\ \alpha_g \rho_g \\ \alpha_g \rho_g v_g \\ \alpha_g \rho_g E_g \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} 0 \\ \alpha_\ell \rho_\ell v_\ell \\ \alpha_\ell(\rho_\ell v_\ell^2 + p_\ell) \\ \alpha_\ell(\rho_\ell E_\ell + p_\ell) v_\ell \\ \alpha_g \rho_g v_g \\ \alpha_g(\rho_g v_g^2 + p_g) \\ \alpha_g(\rho_g E_g + p_g) v_g \end{pmatrix}$$

$$B(\mathbf{u}) = \begin{pmatrix} v_\mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_\mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_\mathrm{i} v_\mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -p_\mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 \\ -p_\mathrm{i} v_\mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{s}(\mathbf{u}) = \begin{pmatrix} \Gamma_\mathrm{i}/\rho_\mathrm{i} \\ -\Gamma_\mathrm{i} \\ -F_\mathrm{i} - v_\mathrm{i}\Gamma_\mathrm{i} \\ -v_\ell F_\mathrm{i} + Q_{\mathrm{i}\,\ell} - E_{\mathrm{i}\,\ell}\Gamma_\mathrm{i} \\ \Gamma_\mathrm{i} \\ F_\mathrm{i} + v_\mathrm{i}\Gamma_\mathrm{i} \\ v_g F_\mathrm{i} + Q_{\mathrm{i}\,g} + E_{\mathrm{i}\,g}\Gamma_\mathrm{i} \end{pmatrix}$$

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Model properties

1. Source terms and interphase quantities
2. Conservation of mass, momentum and energy at the interface
3. Equation of state
   → How to describe pressure p ?
4. Well-posedness of the model
   → Hyperbolicity
5. Entropy inequality
   → Consistent with 2nd law of thermodynamics

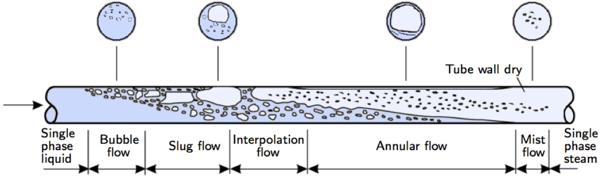
$$\begin{pmatrix} 0 \\ \alpha_\ell \rho_\ell v_\ell \\ \ldots_\ell(\rho_\ell v_\ell^2 + p_\ell) \\ \ldots(\rho_\ell E_\ell + p_\ell) v_\ell \\ \alpha_g \rho_g v_g \\ \ldots_g(\rho_g v_g^2 + p_g) \\ \ldots(\rho_g E_g + p_g) v_g \end{pmatrix}$$

$$\mathbf{s}(\mathbf{u}) = \begin{pmatrix} \Gamma_i/\rho_i \\ -\Gamma_i \\ -F_i - v_i\Gamma_i \\ -v_\ell F_i + Q_{i\,\ell} - E_{i\,\ell}\Gamma_i \\ \Gamma_i \\ F_i + v_i\Gamma_i \\ v_g F_i + Q_{i\,g} + E_{i\,g}\Gamma_i \end{pmatrix}$$

Two-phase flow model | ① Source terms

$$B(\mathbf{u}) = \begin{pmatrix} v_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_i & 0 & 0 & 0 & 0 & 0 & 0 \\ p_i v_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -p_i & 0 & 0 & 0 & 0 & 0 & 0 \\ -p_i v_i & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{s}(\mathbf{u}) = \begin{pmatrix} \Gamma_i/\rho_i \\ -\Gamma_i \\ -F_i - v_i\Gamma_i \\ -v_\ell F_i + Q_{i\ell} - E_{i\ell}\Gamma_i \\ \Gamma_i \\ F_i + v_i\Gamma_i \\ v_g F_i + Q_{ig} + E_{ig}\Gamma_i \end{pmatrix}$$

- Interphase quantities: $\Gamma_i, v_i, p_i, E_{i\ell}, E_{ig}, \rho_i = ???$
- Flow regimes for friction F_i:



- Models for heat transfer $Q_{i\ell}$, Q_{ig}:

  Convection, Condensation, Nucleate & Film boiling

Two-phase flow model | ② Conservation at interface

$$\mathbf{s}(\mathbf{u}) = \begin{pmatrix} \Gamma_\mathrm{i}/\rho_\mathrm{i} \\ -\Gamma_\mathrm{i} \\ -F_\mathrm{i} - v_\mathrm{i}\Gamma_\mathrm{i} \\ -v_\ell F_\mathrm{i} + Q_{\mathrm{i}\,\ell} - E_{\mathrm{i}\,\ell}\Gamma_\mathrm{i} \\ \Gamma_\mathrm{i} \\ F_\mathrm{i} + v_\mathrm{i}\Gamma_\mathrm{i} \\ v_g F_\mathrm{i} + Q_{\mathrm{i}\,g} + E_{\mathrm{i}\,g}\Gamma_\mathrm{i} \end{pmatrix}$$

Heat conduction limited model

$$\Gamma_\mathrm{i} = \frac{1}{E_{\mathrm{i}\,\ell} - E_{\mathrm{i}\,g}}\Big(F_\mathrm{i}(v_g - v_\ell) + Q_{\mathrm{i}\,\ell} + Q_{\mathrm{i}\,g}\Big)$$

RELAP [4].

Two-phase flow model | ③ Equation of state

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} 0 \\ \alpha_\ell \rho_\ell v_\ell \\ \alpha_\ell(\rho_\ell v_\ell^2 + p_\ell) \\ \alpha_\ell(\rho_\ell E_\ell + p_\ell) v_\ell \\ \alpha_g \rho_g v_g \\ \alpha_g(\rho_g v_g^2 + p_g) \\ \alpha_g(\rho_g E_g + p_g) v_g \end{pmatrix}$$

Describe p by two state parameter

$$p_\ell = p(\rho_\ell, u_\ell), \quad p_g = p(\rho_g, u_g)$$

with density ρ and specific inner energy u.

Two-phase flow model | ④ Hyperbolicity

Rewrite system in terms of primitive quantities

$$\partial_t \mathbf{w} + M(\mathbf{w})\partial_x \mathbf{w} = \tilde{\mathbf{s}}(\mathbf{w})$$

Eigenvalues

$$\lambda = \left(v_\mathrm{i}, \quad v_\ell, \quad v_\ell + w_\ell, \quad v_\ell - w_\ell, \quad v_g, \quad v_g + w_g, \quad v_g - w_g\right)^\mathsf{T}$$

with speed of sound w.

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with speed of sound w. Eigenvectors form a basis of $\mathbb{R}^7$ as soon as the **non-resonance condition** is fulfilled Coquel, Hérard, Saleh, and Seguin [5] :

$$v_\mathrm{i} \neq v_\ell \pm w_\ell \quad \text{and} \quad v_\mathrm{i} \neq v_g \pm w_g .$$

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$$v_\mathrm{i} \neq v_\ell \pm w_\ell \quad \text{and} \quad v_\mathrm{i} \neq v_g \pm w_g.$$

Gallouët, Hérard, and Seguin [6] choose v_i as convex combination between v_ℓ and v_g:

$$v_\mathrm{i} := \beta v_\ell + (1-\beta) v_g \quad \text{with} \quad \beta \in [0,1]$$

Non-resonance condition will always be fulfilled
→ M is diagonalisable → quasilinear system is **hyperbolic**.

Two-phase flow model | ⑤ Entropy-entropy flux pair

Closed quasilinear form: $\partial_t \mathbf{u} + A(\mathbf{u}) \cdot \partial_x \mathbf{u} = \mathbf{s}(\mathbf{u})$.

Find entropy function $\eta(\mathbf{u})$ and entropy flux $\psi(\mathbf{u})$, such that

$$\partial_t \eta(\mathbf{u}) + \partial_x \psi(\mathbf{u}) \overset{!}{\leq} 0$$

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with

1. Convex entropy (decreasing behaviour):
$$\eta''(\mathbf{u}) > 0$$

2. Compatibility condition of Tadmor [8]:
$$\partial_{\mathbf{u}} \psi(\mathbf{u})^{\mathsf{T}} \overset{!}{=} \partial_{\mathbf{u}} \eta(\mathbf{u})^{\mathsf{T}} A(\mathbf{u})$$

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$$\partial_{\mathbf{u}} \psi(\mathbf{u})^{\mathsf{T}} \overset{!}{=} \partial_{\mathbf{u}} \eta(\mathbf{u})^{\mathsf{T}} A(\mathbf{u})$$
3. Entropy production condition:
$$\begin{aligned} & \partial_t \mathbf{u} + A(\mathbf{u}) \cdot \partial_x \mathbf{u} = \mathbf{s}(\mathbf{u}) \\ \Leftrightarrow\ & \partial_{\mathbf{u}} \eta(\mathbf{u})^{\mathsf{T}} \partial_t \mathbf{u} + \partial_{\mathbf{u}} \eta(\mathbf{u})^{\mathsf{T}} A(\mathbf{u}) \cdot \partial_x \mathbf{u} = \partial_{\mathbf{u}} \eta(\mathbf{u})^{\mathsf{T}} \mathbf{s}(\mathbf{u}) \\ \Leftrightarrow\ & \partial_t \eta(\mathbf{u}) + \partial_x \psi(\mathbf{u}) = \partial_{\mathbf{u}} \eta(\mathbf{u})^{\mathsf{T}} \mathbf{s}(\mathbf{u}) \overset{!}{\leq} 0 \end{aligned}$$

Two-phase flow model | ⑤ Entropy-entropy flux pair (1)

Choose mixture of physical entropy s_ℓ and s_g:

$\eta(\mathbf{u}) = -(\alpha_\ell \rho_\ell s_\ell + \alpha_g \rho_g s_g)$ and $\psi(\mathbf{u}) = -(\alpha_\ell \rho_\ell s_\ell v_\ell + \alpha_g \rho_g s_g v_g)$

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1. Convexity of $\eta(\mathbf{u})$:

   Follow proof of Coquel, Hérard, Saleh, and Seguin [5], using results of Godlewski and Raviart [7].

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2. Compatibility condition $\partial_{\mathbf{u}}\psi(\mathbf{u})^\mathsf{T} \overset{!}{=} \partial_{\mathbf{u}}\eta(\mathbf{u})^\mathsf{T} \cdot A(\mathbf{u})$:

$$
\begin{pmatrix}
\frac{p_\ell v_\ell}{T_\ell} - \frac{p_g v_g}{T_g} \\
v_\ell \cdot \frac{\frac{p_\ell}{\rho_\ell} + u_\ell - \frac{1}{2}v_\ell^2}{T_\ell} \\
\frac{v_\ell^2}{T_\ell} - s_\ell \\
-\frac{v_\ell}{T_\ell} \\
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\end{pmatrix}
\overset{!}{=}
\begin{pmatrix}
\frac{p_\ell v_\ell}{T_\ell} - \frac{p_g v_g}{T_g} - \frac{(p_\ell - p_\mathrm{i})(v_\ell - v_\mathrm{i})}{T_\ell} + \frac{(p_g - p_\mathrm{i})(v_g - v_\mathrm{i})}{T_g} \\
v_\ell \cdot \frac{\frac{p_\ell}{\rho_\ell} + u_\ell - \frac{1}{2}v_\ell^2}{T_\ell} \\
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\end{pmatrix}
$$

Two-phase flow model | ⑤ Entropy-entropy flux pair (2)

Choose mixture of physical entropy s_ℓ and s_g:

$$\eta(\mathbf{u}) = -(\alpha_\ell \rho_\ell s_\ell + \alpha_g \rho_g s_g) \text{ and } \psi(\mathbf{u}) = -(\alpha_\ell \rho_\ell s_\ell v_\ell + \alpha_g \rho_g s_g v_g)$$

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Interphasic velocity *[Hyperbolicity]*

$$v_\mathrm{i} = \beta v_\ell + (1-\beta) v_g \quad \text{with } \beta \in [0,1]$$

Interphasic pressure Gallouët, Hérard, and Seguin [6]

$$p_\mathrm{i} := \gamma p_\ell + (1-\gamma) p_g \quad \text{with} \quad \gamma \in [0,1]$$

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Interphasic velocity *[Hyperbolicity]*

$$v_{\mathrm{i}} = \beta v_\ell + (1-\beta) v_g \quad \text{with } \beta \in [0,1]$$

Interphasic pressure

$$p_{\mathrm{i}} := \gamma p_\ell + (1-\gamma) p_g \quad \Rightarrow \quad \gamma = \frac{(1-\beta) T_g}{\beta T_\ell + (1-\beta) T_g}$$

Two-phase flow model | ⑤ Entropy-entropy flux pair (3)

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$\eta(\mathbf{u}) = -(\alpha_\ell \rho_\ell s_\ell + \alpha_g \rho_g s_g)$ and $\psi(\mathbf{u}) = -(\alpha_\ell \rho_\ell s_\ell v_\ell + \alpha_g \rho_g s_g v_g)$

3. Entropy inequality of entropy production: $\partial_{\mathbf{u}} \eta(\mathbf{u})^{\mathsf{T}} \mathbf{s}(\mathbf{u}) \overset{!}{\leq} 0$:

$$- \frac{Q_{\mathrm{i}\,\ell}}{T_\ell} + \frac{E_{\mathrm{i}\,\ell} - E_\ell + v_\ell^2 - v_\ell v_{\mathrm{i}} + \frac{p_\ell}{\rho_{\mathrm{i}}} - \frac{p_\ell}{\rho_\ell} + s_\ell T_\ell}{T_\ell} \cdot \Gamma_{\mathrm{i}}$$

$$- \frac{Q_{\mathrm{i}\,g}}{T_g} - \frac{E_{\mathrm{i}\,g} - E_g + v_g^2 - v_g v_{\mathrm{i}} + \frac{p_g}{\rho_{\mathrm{i}}} - \frac{p_g}{\rho_g} + s_g T_g}{T_g} \cdot \Gamma_{\mathrm{i}} \overset{!}{\leq} 0$$

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$$-\frac{Q_{\mathrm{i}\,g}}{T_g} - \frac{E_{\mathrm{i}\,g} - E_g + v_g^2 - v_g v_{\mathrm{i}} + \frac{p_g}{\rho_{\mathrm{i}}} - \frac{p_g}{\rho_g} + s_g T_g}{T_g} \cdot \Gamma_{\mathrm{i}} \overset{!}{\leq} 0$$

Spec. total energy = spec. enthalpy − spec. pressure + kinetic energy (physical law)

$$E_\ell = h_\ell - \frac{p_\ell}{\rho_\ell} + \frac{1}{2} v_\ell^2, \qquad E_{\mathrm{i}\,\ell} := h_{\ell\,\mathrm{sat}} - \frac{p_{\mathrm{i}}}{\rho_{\mathrm{i}}} + \frac{1}{2} v_{\mathrm{i}}^2$$

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Two-phase flow model | ⑤ Entropy-entropy flux pair (3)

Choose mixture of physical entropy s_ℓ and s_g:

$\eta(\mathbf{u}) = -(\alpha_\ell \rho_\ell s_\ell + \alpha_g \rho_g s_g)$ and $\psi(\mathbf{u}) = -(\alpha_\ell \rho_\ell s_\ell v_\ell + \alpha_g \rho_g s_g v_g)$

3 Entropy inequality of entropy production: $\partial_{\mathbf{u}} \eta(\mathbf{u})^T \mathbf{s}(\mathbf{u}) \overset{!}{\leq} 0$:

$$-\frac{Q_{i\,\ell}}{T_\ell} + \frac{h_{\ell\,\text{sat}} - h_\ell + \frac{1}{2}(v_\ell - v_i)^2 + \frac{p_\ell - p_i}{\rho_i} + s_\ell T_\ell}{T_\ell} \cdot \Gamma_i$$

$$-\frac{Q_{i\,g}}{T_g} - \frac{h_{g\,\text{sat}} - h_g + \frac{1}{2}(v_g - v_i)^2 + \frac{p_g - p_i}{\rho_i} + s_g T_g}{T_g} \cdot \Gamma_i \overset{!}{\leq} 0$$

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Interphasic velocity

$$v_i := \beta v_\ell + (1-\beta) v_g \quad \text{with} \quad \beta \in [0,1]$$

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Interphasic density

$$\rho_i := \rho_{\text{sat}}$$

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$$- \frac{Q_{i\,g}}{T_g} - \frac{h_{g\,\text{sat}} - h_g + \frac{p_g - p_i}{\rho_i} + s_g T_g}{T_g} \cdot \Gamma_i \overset{!}{\leq} 0$$

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$$\rho_i := \rho_{\text{sat}}$$

→ Check entropy inequality within physical relevant region!

Two-phase flow model | Summary

Model properties

- ✓ Source terms and interphase quantities
- ✓ Conservation of mass, momentum and energy at the interface
- ✓ Equation of state
- ✓ Well-posed hyperbolic model
- ✓ Entropy-Entropy flux pair
  Consistent with 2nd law of thermodynamics

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- ✓ Source terms and interphase quantities
- ✓ Conservation of mass, momentum and energy at the interface
- ✓ Equation of state
- ✓ Well-posed hyperbolic model
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  Consistent with 2nd law of thermodynamics
- – No linear degenerated field for first eigenvalue $\lambda_1 = v_\mathrm{i}$,
  only iff $v_\mathrm{i} := v_\ell$, or $v_\mathrm{i} := v_g$, or $v_\mathrm{i} := \frac{\alpha_\ell \rho_\ell v_\ell + \alpha_g \rho_g v_g}{\alpha_\ell \rho_\ell + \alpha_g \rho_g}$.

Two-phase flow model | Next steps

Numerical schemes for quasilinear system

1. Path-conservative scheme, Castro et al. [10]
   - transform into homogeneous system in non-conservative form
   - path connects two states $\mathbf{u}_L$ and $\mathbf{u}_R$ at its left x_L and right x_R limits across a discontinuity
2. Relaxation, Baudin, Berthon, Coquel, Masson, and Tran [11]
   - transform system, such that it is linearly degenerated (?)
   - extend system with relaxed pressure and temperature equations
   - system linearly degenerate $\rightarrow$ easy to find Riemann solution.

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