EGRIN
Ecoulements Gravitaires et RIsques Naturels
Modeling and Numerics
for (Shallow) (Water) Flows

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GdR EGRIN
http ://gdr-egrin.math.cnrs.fr/

May 29, 2017

EGRIN : Teams

- (Applied) Mathematics
  Paris-Est, Paris-Nord, Paris-Sud, UPMC, Dauphine, Descartes, Nantes, Clermont, Besançon, Montpeliier, Rennes, Bordeuax, Lyon, Nice, Chambery, Orléans, Toulouse, Amiens, Vannes, Toulon, Grenoble, Marseille, Corse, Versailles, Seville
- Physics & Mechanics
  IPGP, IPR, LISAH, IMFT, LTHE, LGGE, ISTO, LMV, LMD, IUSTI, IJRA, Navier, HSM, ETNA, LOF, LPMC, PMMD
- INRIA
  ANGE, LEMON, AIRSEA, CARDAMOM
- State Institutes & Companies
  CEREMA, CERFACS, BRGM, INRA, EDF, ANTEA, LHSV

$\rightsquigarrow \approx$ 250 members

EGRIN : Board

- C. Lucas (MAPMO, Orléans)
- J. Sainte-Marie (CEREMA, ANGE Team)
- C. Berthon (LJL, Nantes)
- F. Bouchut (LAMA, Paris Est)
- L. Chupin (LM, Clermont)
- S. Cordier (MAPMO, Orléans)
- A. Mangeney (IPGP, Paris)
- P. Saramito (LJK, Grenoble)
- A. Valance (IPR, Rennes & GdR TransNat)
- S. Da Veiga (SAFRAN & GdR MascotNum)

EGRIN : Collaborations

- GdR TransNat
  http://transnat.univ-rennes1.fr/
- GdR MePhy
  Mécanique et Physique des Systèmes Complexes
  https://www.pmmh.espci.fr/ mephy/wiki/doku.php?id=start
- GdR Films
  Ruissellement et films cisaillés
  https://www.pmmh.espci.fr/ mephy/wiki/doku.php?id=start
- GdR Ma-Nu
  Mathématiques pour le Nucléaire
  http://gdr-manu.math.cnrs.fr/
- GdR Mascot-Num
  Analyse Stochastique pour Codes et Traitements Numériques
  http://www.gdr-mascotnum.fr/

EGRIN : Annual Workshops

- Orléans 2013 & 2014
  - R. Delannay (Rennes) : Granular flows
  - J. Garnier (Diderot) : Rare events simulations
  - A. Valance (Rennes) : Dune dynamics
  - E. Blayo (INRIA Grenoble) : Data assimilation
  - E. Fernandez Nieto (Séville) : High Order Finite Volumes
  - P.Y. Lagrée (UPMC) : Hydrodynamics & Erosion models
- Nantes 2015 & 2016
  - D. Gérard Varet (Diderot) : Rough boundaries effects
  - N. Mangold (Nantes) : Gravity driven flows on planets
  - P. Saramito (Grenoble) : Numerics for Viscoplastic fluids
  - A.L. Dalibard (UPMC) : Primitive equations of the ocean
  - T. Lelievre (CERMICS) : Model Reduction Technics
  - O. Roche (Clermont) : Fluidized Granular Flows
- Cargese 2017
  - P. Bonneton (Bordeaux) : Dispersive waves
  - F. Bouchut (UMLV) : Numerical methods for complex rheology
  - A. Mangeney (IPGP) : Granular geophysical flows

EGRIN : Thematics

Le programme scientifique de EGRIN est centré sur la formulation et la résolution numérique de modèles de complexité réduite par rapport aux équations de Navier-Stokes à surface libre, mais s'affranchissant des hypothèses trop restricitives qu'on retrouve dans les modèles classiques d'écoulements peu profonds. [...]

On s'intéresse aux écoulements complexes et aux couplages induits lorsque le fluide interagit avec les sols ou les structures (érosion, glissements de terrains...). Les fluides considérés sont eux-mêmes complexes, au sens où ils possèdent une rhéologie particulière (avalanches, écoulements pyroclastiques...). [...]

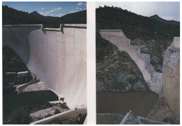
Sur ces sujets, il est nécessaire de décloisonner les disciplines et les mathématiciens doivent tisser des liens avec des modélisateurs non-mathématiciens pour mieux prendre en compte les problèmes.

EGRIN : Thematics

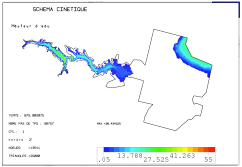
- Hydrodynamics
  $\rightsquigarrow$ Tsunamis, Flooding, Dam breaks, Rogue waves...
- Complex Fluids
  $\rightsquigarrow$ Avalanches, Mud or Pyroclastic flows, Multiphase flows...
- Coupling
  $\rightsquigarrow$ Morphodynamics, Biological phenomena, Fluid-Structure...
- Modeling
  $\rightsquigarrow$ Conservation, Energy inequality, Well-posedness...
- Numerical Analysis
  $\rightsquigarrow$ Accuracy, Robustness, Non linear Stability, WB...
- Data Assimilation
  $\rightsquigarrow$ Paramester estimation, Filtering, Control...

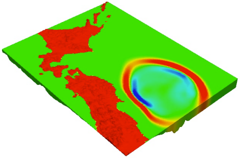
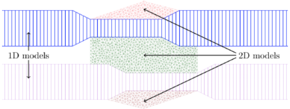
EGRIN : Basics

- (Natural ?) hazards



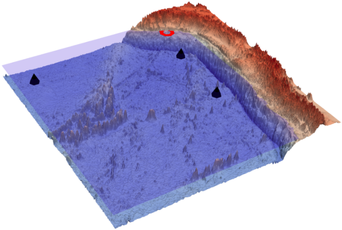
- Simulations with Shallow Water Flows

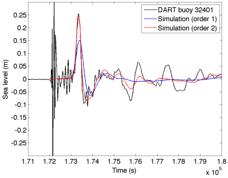


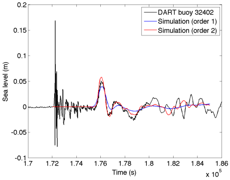



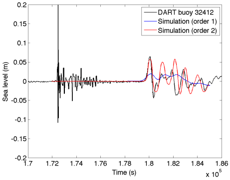
EGIN : Basics

- Comparisons with Experiments & Measurements



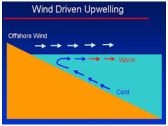


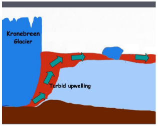




EGRIN : Main Focuses

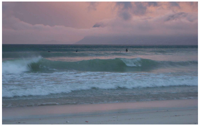
- Hydrostatic NS equations : Multilayer models (ABPS [11], ABPSM [11], Rambaud [12]...)





- Non Hydrostatic SW Models : Boussinesq type models (Bonneton et al. [11], Sainte-Marie [11]...)





EGRIN : Main Focuses

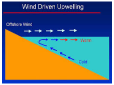
- Complex flows : Generalized topography, Alternative rheology (Mangeney et al. [03], Bouchut-Westdickenberg [04], Chupin [09], Nieto et al. [10]...)

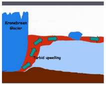

- Erosion processes, Sediment transport (Castro et al. [08], Bouharguane-Mohammadi [09], Cordier et al. [11], ABCDGGJSS [11]...)



EGRIN : Ongoing Work

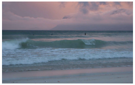
- Hydrostatic NS equations : Multilayer models (Parisot - Vila [14], Di Martino - Haspot [17], Couderc - Duran - Vila [17], AAGP [17]...)





- Non Hydrostatic SW Models : Boussinesq type models (Duran - Marche [15], Lannes - Marche [15], Bristeau et al. [15], Aissiouene [16]...)





⇝ Non Hydrostatic NS Equations : Layerwise models (Fernandez Nieto - Parisot - Penel - Sainte Marie [17])

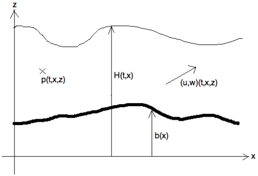
EGRIN : Ongoing Work

- Complex flows
  (Gueugneau-Chupin et al. [17], Bouchut - Mangeney et al. [16], Saramito - Wachs [17], Bristeau et al. [17], Fernandez - Gallardo - Vigneaux [17]...)


- Erosion processes, Sediment transport
  (Fernandez - Narbona et al. [16], Mohamadi [16], Nouhou Bako et al. [17], ABP [17])



Free Surface Incompressible Navier Stokes Equations



- Computational domain

$$z \in [b(x), H(t,x)]$$

- Equations

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{g} + \nabla \cdot \sigma_v \end{aligned}$$

- Boundary conditions

Shallow Water Equations

- 1d shallow water equations

$$\begin{aligned} \partial_t h + \partial_x hu &= 0, \\ \partial_t hu + \partial_x \left(hu^2 + gh^2/2\right) &= -gh\,\partial_x b + S_f \end{aligned}$$

with
h : water depth, b : bottom topography,
u : velocity of the water column, S_f : friction term

Saint-Venant (1871),
"Théorie du mouvement non-permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit", Comptes Rendus Acad. Sciences.

Shallow Water Equations : Derivation (SV1)

- Derivation

6. *Théorie et équations générales du mouvenant non permanent des eaux courantes.* — Cette théorie, dont le Mémoire que m'a communiqué M. Partiot m'a donné l'occasion de faire l'étude, peut être présentée d'une manière directe et générale, sans invoquer la formule de propagation des ondes ou des intumescences due à Lagrange.

Soit en effet, à l'époque du temps quelconque t, dans un canal ou une rivière :

U la vitesse moyenne des eaux à travers une section transversale ω, dont s est l'abscisse, sensiblement horizontale, comptée à partir d'un endroit quelconque de son cours.

- Mass budget

1^re^ *équation.* Le volume ωds d'une couche comprise entre les deux sections ayant pour abscisses

$$s \quad \text{et} \quad s + ds$$

ne doit pas être changé lorsque, après un petit temps Δt, ces abscisses sont devenues

$$s + \mathrm{U}\Delta t \quad \text{et} \quad s + ds + \left(\mathrm{U} + \frac{d\mathrm{U}}{ds} ds\right)\Delta t,$$

et que son épaisseur, ainsi, est devenue

$$ds + \frac{d\mathrm{U}}{ds} ds \Delta t;$$

or, alors, la section a pour superficie

$$\omega + \left(\frac{d\omega}{dt} + \frac{d\omega}{ds}\frac{ds}{dt}\right)\Delta t = \omega + \frac{d\omega}{dt}\Delta t + \frac{d\omega}{ds}\mathrm{U}\Delta t.$$

Égalant le produit de cette superficie par cette épaisseur, à ωds, réduisant et divisant par $\Delta t ds$, on obtient, pour la condition de continuité ou de conservation du volume,

$$\frac{d\omega}{dt} + \frac{d(\omega \mathrm{U})}{ds} = 0. \qquad (19)$$

- Gravity tem

» Quantité de mouvement due à la pesanteur,

$$\rho g \omega ds \frac{d(\zeta + y)}{ds} \Delta t;$$

- Pressure term

» Quantité de mouvement due aux pressions sur deux faces opposées ay et $a\left(y + \frac{dy}{ds} ds\right)$ de la tranche,

$$\rho g a \left[y \frac{y}{2} - \frac{1}{2}\left(y + \frac{dy}{ds} ds\right)^2 \right] dt = -\rho g \omega ds \frac{dy}{ds} \Delta t;$$

- Friction term

» Quantité de mouvement due au frottement ou à la résistance tangentielle du fond,

$$-\rho g F \chi ds;$$

- Momentum budget

» Égalant la somme des trois premières quantités à la troisième, et divisant tout par $\rho g \omega ds \Delta t$, on a la deuxième équation

$$(20) \qquad \frac{d\zeta}{ds} = \frac{1}{g} \frac{dU}{dt} + \frac{U}{g} \frac{dU}{ds} + \frac{\chi}{\omega} \frac{F}{\rho g}.$$

Shallow Water Equations : Derivation (SV1)

- Hypothesis : Almost flat bottom

» Si nous estimons les quantités de mouvement parallèlement au fond supposé très-peu incliné, nous pourrons prendre pour les grandeurs des composantes de la vitesse de la tranche et des pressions qu'elle supporte, celles de cette vitesse et de ces pressions elles-mêmes, car elles n'en diffèrent que de quantités négligeables. On aura donc :

- Hypothesis : Constant velocity

» D'autres intumescences viennent ensuite se superposer à celle-ci. L'eau qu'elles introduisent, se mêlant à celle du canal, lui fait acquérir une certaine *vitesse* horizontale que nous supposerons, en abstrayant les frottements, sensiblement la même de la surface au fond. Si nous appelons U cette vitesse acquise quand la profondeur d'eau est devenue y, l'intumes–

- Hypothesis : Rectangular channel

2[e] *équation*. Exprimons l'équilibre dynamique, pour le temps Δt, des quantités de mouvement de la même tranche, décomposées ou estimées dans une direction unique, en nous bornant à un canal à section rectangulaire de profondeur variable y et de largeur constante a, d'où $\omega = ay$.

- Conclusion

» Je donnerai, dans un autre article, l'intégration de ces équations pour un cas étendu. »

Shallow Water Equations : Properties

- 2d shallow water equations with sources

$$\begin{aligned} \partial_t h + \nabla \cdot (h\bar{\mathbf{u}}) &= 0, \\ \partial_t (h\bar{\mathbf{u}}) + \nabla \cdot \left(h\bar{\mathbf{u}} \otimes \bar{\mathbf{u}} + \frac{gh^2}{2} I \right) &= -gh\nabla b - 2\Omega \times h\bar{\mathbf{u}} - \kappa(h, \bar{\mathbf{u}})\bar{\mathbf{u}} \end{aligned}$$

- Properties
  - Conservation law
  - Hyperbolic system (wave propagation, weak solution)
  - Positivity of water depth (invariant domain, dry zones)
  - Energy (entropy) (in)equality
  - Non-trivial steady states (lake at rest)

Shallow Water Equations : Numerics (Finite Volumes)

- First Works
  - Bermudez-Vazquez ['94], Greenberg-Leroux ['96]
  - Goutal-Maurel ['97]
- Positive and well-balanced numerical schemes ['01 $\rightarrow$ '16]
  - Extended Godunov scheme (Chinnayya-Leroux-Seguin)
  - Kinetic interpretation of source term (Perth.-Sim.,ABBSM)
  - Extended Suliciu relaxation scheme v1 (Bouchut,Galice,ACU)
  - Hydrostatic reconstruction (ABBKP, Liang-Marche)
  - Path-conservative scheme (Castro-Macias-Pares)
  - Hydrostatic upwind scheme (Berthon-Foucher)
  - Central scheme (Kurganov, Kurganov-Noelle)
- Review books
  - Hyperbolic Problems & Finite Volumes Method
    Godlewski-Raviart ['96], Toro ['99], Leveque ['02]...
  - Bouchut ['04]
    Nonlinear Stability of FV Methods for Hyperbolic
    Conservations Laws and WB Schemes for Sources, Birkhäuser.

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Shallow Water Equations : Derivation (SV2)

- Navier-Stokes Equations
- Small aspect ratio : $H/L = \epsilon$
  - Shallow flows
  - Long waves
- Rescaled Viscosity, Friction, Bottom and Free Surface Slopes

$\rightsquigarrow$ Hydrostatic Pressure Distribution (at 1^{st} order)

$\rightsquigarrow$ Constant vertical profile for horizontal velocity (at 1^{st} order)

$\rightsquigarrow$ Parabolic vertical profile for horizontal velocity (at 2^{nd} order)

$\rightsquigarrow$ "Viscous" Shallow Water Equations on $(h, \bar{u})$

Gerbeau & Perthame, M2AN ['01]

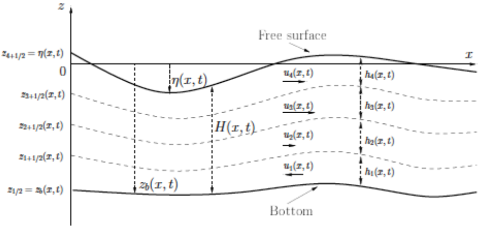
Ferrari-Saleri [04], Marche [07], Decoene [08], Frings [12]

Layerwise Models : Basic case

- Euler Equations
- Hydrostatic Pressure

⇝ Piecewise constant vertical profile for horizontal velocity

$$u(t,x,z) = \sum_{\alpha=1}^{M} 1_{[z_{\alpha-1/2}(t,x), z_{\alpha+1/2}(t,x)]} \bar{u}_\alpha(t,x)$$



ABPS, M2AN ['11]

Layerwise Models : Basic case

- SW type model
  - No SW hypothesis !
  - Arbitrary shalowness of the "layers"

$$\partial_t H + \partial_x H\bar{u} = 0$$

$$\partial_t H\bar{u}_\alpha + \partial_x H\bar{u_\alpha}^2 + \partial_x gH^2/2$$
$$= -gH\partial_x b + u_{\alpha+1/2}G_{\alpha+1/2} - u_{\alpha-1/2}G_{\alpha-1/2}$$

$$z_{\alpha+1/2}(t,x) = b(x) + \sum_{\beta=1}^{\alpha} l_\beta H(t,x)$$

- Closures for $u_{\alpha+1/2}$ and $G_{\alpha+1/2}$
- Extension of FV techniques
  - ⇝ Conservation
  - ⇝ Non linear Stability
  - ⇝ Well Balancing
  - ⇝ Non connex domains

ABPS, M2AN ['11]

Layerwise Models : Variable Density case

- Euler Equations
- Hydrostatic Pressure
- Density is function of tracers

$\rightsquigarrow$ Piecewise constant vertical profile for tracers

$$T(t,x,z) = \sum_{\alpha=1}^{M} 1_{[z_{\alpha-1/2}, z_{\alpha+1/2}]} \bar{T}_\alpha(t,x)$$

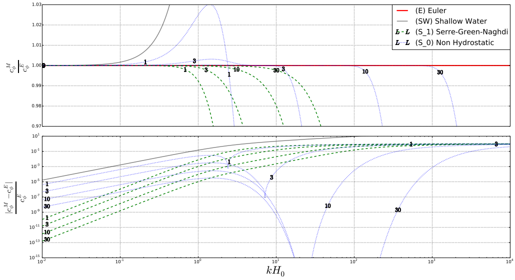
ABPS, JCP ['11]

Layerwise Models : Non hydrostatic Version

- Euler Equations
- Non Hydrostatic Pressure

⇝ Piecewise linear vertical profile for vertical velocity

$$w(t,x,z) = \sum_{\alpha=1}^{M} 1_\alpha \left(\bar{w}_\alpha(t,x) + 2\sqrt{3}\sigma_\alpha(t,x)(z - z_\alpha(t,x))/l_\alpha \right)$$



Fernandez Nieto - Parisot - Penel - Sainte Marie ['17]

Layerwise Models : Non hydrostatic Version

$$\partial_t H + \partial_x (H\bar{u}) = 0,$$

$$\partial_t(h_\alpha \bar{u_\alpha}) + \partial_x \left(h_\alpha \bar{u_\alpha}^2 + h_\alpha \bar{q}_\alpha\right) + \left(u_{\alpha+1/2}\Gamma_{\alpha+1/2} - \partial_x z_{\alpha+1/2} q_{\alpha+1/2}\right) - [.]_{\alpha-1/2} = -h_\alpha \partial_x (g\eta + p^{atm}),$$

$$\partial_t(h_\alpha \bar{w}_\alpha) + \partial_x (h_\alpha \bar{u_\alpha} \bar{w}_\alpha) + \left(w_{\alpha+1/2}\Gamma_{\alpha+1/2} + q_{\alpha+1/2}\right) - [.]_{\alpha-1/2} = 0,$$

$$\partial_t(h_\alpha \sigma_\alpha) + \partial_x(h_\alpha \sigma_\alpha \bar{u_\alpha}) = 2\sqrt{3}\left[\bar{q}_\alpha - \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} - \Gamma_{\alpha+1/2}\left(\frac{h_\alpha \partial_x \bar{u_\alpha}}{12} + \frac{w_{\alpha+1/2} - \bar{w}_\alpha}{2}\right) + [.]_{\alpha-1/2}\right]$$

$$\partial_x \bar{u_\alpha} + \frac{w^-_{\alpha+1/2} - \bar{w}_\alpha}{h_\alpha/2} = 0, \qquad \sigma_\alpha + \frac{h_\alpha \partial_x \bar{u_\alpha}}{2\sqrt{3}} = 0,$$

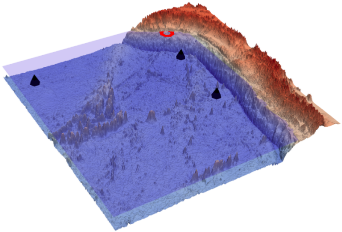
$$w^+_{\alpha+1/2} - \partial_t z_b - \bar{u}_{\alpha+1}\partial_x z_{\alpha+1/2} + \sum_{\beta=1}^{\alpha} \partial_x (h_\beta \bar{u}_\beta) = 0,$$

$$w^+_{\alpha+1/2} - w^-_{\alpha+1/2} = \partial_x z_{\alpha+1/2}(\bar{u}_{\alpha+1} - \bar{u_\alpha}),$$

Layerwise Models : Non hydrostatic - Monolayer case

- Green-Naghdi type equations
- Mixed Hyperbolic-Elliptic system

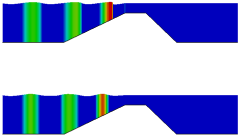
$\rightsquigarrow$ Prediction-Correction Scheme

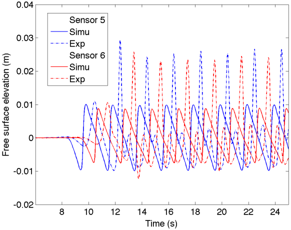


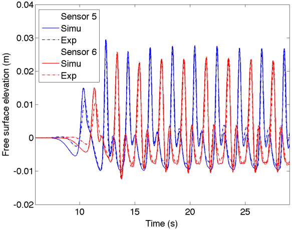
Bristeau - Mangeney - Sainte Marie - Seguin ['15], Aissiouene ['16]

Layerwise Models : Non hydrostatic - Monolayer case

- Dinguemans Experiment







Layerwise Models : With Rheology

- Navier-Stokes Equations
- Hydrostatic Pressure

$\rightsquigarrow$ Piecewise constant vertical profile for stress tensor comp.

$$\frac{\partial}{\partial t}\sum_{j=1}^{N} h_j + \frac{\partial}{\partial x}\sum_{j=1}^{N} h_j u_j = 0,$$

$$\frac{\partial}{\partial t}(h_\alpha u_\alpha) + \frac{\partial}{\partial x}\left(h_\alpha u_\alpha^2 + \frac{g}{2}h_\alpha H\right) = -g h_\alpha \frac{\partial b}{\partial x} + [uG]_\alpha$$
$$+\frac{\partial}{\partial x}\left(h_\alpha \Sigma_{xx,\alpha} - h_\alpha \Sigma_{zz,\alpha} + \frac{\partial}{\partial x}\Big(h_\alpha z_\alpha \Sigma_{zx,\alpha}\Big)\right)$$
$$+z_{\alpha+1/2}\frac{\partial^2}{\partial x^2}\sum_{j=\alpha+1}^{N} h_j \Sigma_{zx,j} - z_{\alpha-1/2}\frac{\partial^2}{\partial x^2}\sum_{j=\alpha}^{N} h_j \Sigma_{zx,j}$$
$$+\sigma_{\alpha+1/2} - \sigma_{\alpha-1/2}$$

Bristeau - Guichard - Di Martino - Sainte Marie ['17]

Viscoplastic flows : Static - Flowing Interface

- Yield Stress (Drücker - Prager)

$$\sigma_v = 2\nu Du + \kappa \frac{Du}{\|Du\|}, \qquad \kappa = \mu_s p_+$$

- Plug zones

$$Du = 0 \qquad (\|\sigma_v\| \leq \kappa)$$

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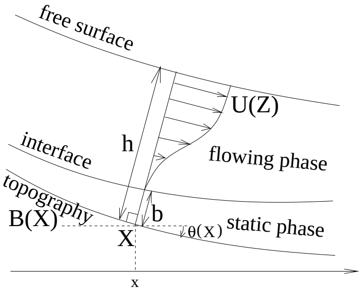


Viscoplastic flows : Static - Flowing Interface

- BCRE Model

$$\partial_t b = \frac{g \cos\theta(\mu_s - |\tan\theta|)}{\partial_z u}$$

- Assumptions
  - Shallow flow
  - Horizontal velocity linear in the vertical coordinate
  - Additional equation to determine $\partial_z u$
  - Hydrostatic Pressure
  - No viscosity

Iverson & Ouyang, Rev. Geophysics ['15]

Viscoplastic flows : Static - Flowing Interface

- BCRE Model

$$\partial_t b = \frac{g \cos\theta(\mu_s - |\tan\theta|)}{\partial_z u}$$

- Assumptions
  - Shallow flow (and small slope)
  - No hypothesis on the velocity profile
  - Classical PDEs on u
  - Non Hydrostatic Pressure (assumed to be convex)
  - Small viscosity

Bouchut - Ionescu - Mangeney ['16]

Viscoplastic flows : Static - Flowing Interface

- PDEs of the new Model

$$\partial_t \left(h - \frac{h^2}{2} d_X \theta \right) + \partial_X \left(\int_0^h U dZ \right) = 0,$$
$$\partial_t U + S - \partial_Z(\nu \partial_Z U) = 0 \qquad \text{for } Z > b(t,X),$$

- Algebraic Relations

$$S = g(\sin\theta + \partial_X(h\cos\theta)) - \partial_Z(\mu_s p),$$
$$p = g\left(\cos\theta + \sin\theta \partial_X h - 2|\sin\theta| \frac{\partial_X U}{|\partial_Z U|} \right) \times (h - Z)$$

- Boundary Conditions

$$Z = h(t,X) \quad : \quad \nu \partial_Z U = 0$$
$$Z = b(t,X) \quad : \quad U = 0 \quad \text{AND} \quad \nu \partial_Z U = 0$$

- Static Equilibrium Condition

$$S(t,X,b(t,X)) \geq 0$$

Viscoplastic flows : Static - Flowing Interface

- BCRE Model

$$\partial_t b = \frac{g \cos\theta(\mu_s - |\tan\theta|)}{\partial_z u}$$

- New Model with Null viscosity

$$\partial_t b = \frac{S(t,x,b(t,x))}{\partial_z u(t,x,b(t,x))}$$

$\rightsquigarrow$ Equivalent under Hydrostatic pressure assumption

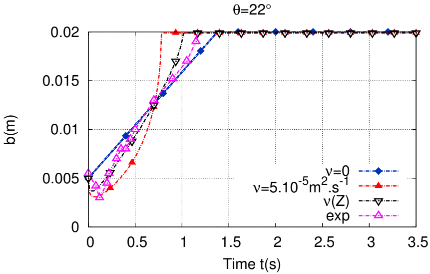
- New Model with viscosity

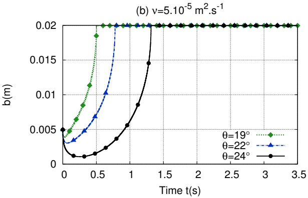
$$\partial_t b = \nu \left(\frac{\partial_z S - \nu \partial^3_{zzz} u}{S} \right)_{z=b}$$

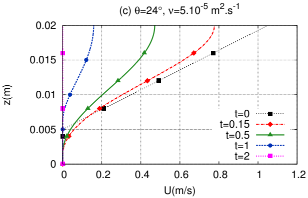
Bouchut - Ionescu - Mangeney ['16]

Martin - Ionecu - Mangeney - Bouchut - Farin ['16]

- 1D-z Simplified Model vs. Experimental Data







Lusso - Bouchut - Ern - Mangeney, ['16]

Multiphase flows

- 3D Jackson Model
  - Solid phase

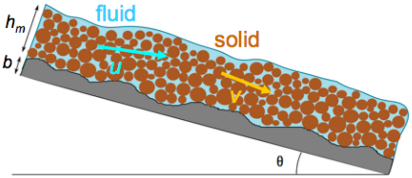
$$\begin{aligned}\partial_t(\rho_s\varphi) + \nabla\cdot(\rho_s\varphi v) &= 0\\ \rho_s\varphi(\partial_t v + (v\cdot\nabla)v) &= -\nabla\cdot T_s + f_0 + \rho_s\varphi\,\mathbf{g}\end{aligned}$$

  - Fluid phase

$$\begin{aligned}\partial_t(\rho_f(1-\varphi)) + \nabla\cdot(\rho_f(1-\varphi)u) &= 0\\ \rho_f(1-\varphi)(\partial_t u + (u\cdot\nabla)u) &= -\nabla\cdot T_{f_m} - f_0 + \rho_f(1-\varphi)\mathbf{g}\end{aligned}$$

$\rightsquigarrow$ Need for closure

Anderson - Jackson, IECM ['67]



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  $\rightsquigarrow$ Need for closure

Anderson - Jackson, IECM ['67]

- 2D Pitman & Le Model
  - No additional equation in the volume
  - Two dynamic boundary conditions at the free surface
  - Shallow layer model
  - No energy

Pitman - Le, Phil. Trans. Roy. Soc. London A ['05]

Multiphase flows

- 3D Jackson Model
  - Solid phase

$$\begin{aligned}\partial_t(\rho_s\varphi) + \nabla\cdot(\rho_s\varphi v) &= 0\\ \rho_s\varphi(\partial_t v + (v\cdot\nabla)v) &= -\nabla\cdot T_s + f_0 + \rho_s\varphi\,\mathbf{g}\end{aligned}$$

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$$\begin{aligned}\partial_t(\rho_f(1-\varphi)) + \nabla\cdot(\rho_f(1-\varphi)u) &= 0\\ \rho_f(1-\varphi)(\partial_t u + (u\cdot\nabla)u) &= -\nabla\cdot T_{f_m} - f_0 + \rho_f(1-\varphi)\mathbf{g}\end{aligned}$$

$\rightsquigarrow$ Need for closure

Anderson - Jackson, IECM ['67]

- 2D New Model
  - Divergence constraint for the solid phase
  - One dynamic boundary condition for the mixture
  - Shallow layer model
  - Energy

Bouchut - Fernandez Nieto - Mangeney - Narbona Reina, M2AN ['15]

Multiphase flows

- 3D New Model
  - Solid phase

$$\begin{aligned} \partial_t(\rho_s\varphi) + \nabla\cdot(\rho_s\varphi v) &= 0 \\ \rho_s\varphi(\partial_t v + (v\cdot\nabla)v) &= -\nabla\cdot T_s + f_0 + \rho_s\varphi\,\mathbf{g} \end{aligned}$$

  - Fluid phase

$$\begin{aligned} \partial_t(\rho_f(1-\varphi)) + \nabla\cdot(\rho_f(1-\varphi)u) &= 0 \\ \rho_f(1-\varphi)(\partial_t u + (u\cdot\nabla)u) &= -\nabla\cdot T_{f_m} - f_0 + \rho_f(1-\varphi)\mathbf{g} \end{aligned}$$

  - Divergence constraint

$$\nabla\cdot v = 0$$

  - Boundary conditions
    - At the free surface : Continuity of the total stress tensor
    - At the bottom : Coulomb friction law + Navier friction

Bouchut - Fernandez Nieto - Mangeney - Narbona Reina, M2AN ['15]

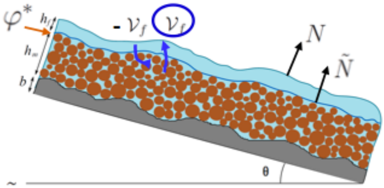
Multiphase flows

- 3D New Model with Dilatancy
  - Solid phase

$$\begin{aligned} \partial_t(\rho_s\varphi) + \nabla\cdot(\rho_s\varphi v) &= 0 \\ \rho_s\varphi(\partial_t v + (v\cdot\nabla)v) &= -\nabla\cdot T_s + f_0 + \rho_s\varphi\,\mathbf{g} \end{aligned}$$

  - Fluid phase

$$\begin{aligned} \partial_t(\rho_f(1-\varphi)) + \nabla\cdot(\rho_f(1-\varphi)u) &= 0 \\ \rho_f(1-\varphi)(\partial_t u + (u\cdot\nabla)u) &= -\nabla\cdot T_{f_m} - f_0 + \rho_f(1-\varphi)\mathbf{g} \end{aligned}$$



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  - Fluid phase

$$\begin{aligned} \partial_t(\rho_f(1-\varphi)) + \nabla\cdot(\rho_f(1-\varphi)u) &= 0 \\ \rho_f(1-\varphi)(\partial_t u + (u\cdot\nabla)u) &= -\nabla\cdot T_{f_m} - f_0 + \rho_f(1-\varphi)\mathbf{g} \end{aligned}$$

  - Divergence equation

$$\nabla\cdot v = \Phi$$

  - Boundary conditions
    - At the free surface : Continuity of the fluid stress tensor
    - At the "solid surface" : Navier friction for fluid + Jump rel.
    - At the bottom : Coulomb friction law + Navier friction

Bouchut - Fernandez Nieto - Mangeney - Narbona Reina, JFM ['16]

Multiphase flows

- 2D Shallow Layer Model with Dilatancy
  - Buoyancy and Drag forces

$$f_0 = -\varphi \nabla p_{f_m} + \beta(u - v)$$

  - Dilatancy

$$\Phi = K\hat{\gamma}(\phi - \phi_c^{eq})$$

  - Asymptotic regimes

$$\beta = \mathcal{O}(\epsilon^{-1}) \quad \text{or} \quad \beta = \mathcal{O}(1)$$

  - Energy inequality
  - Numerical comparisons with
    - Pailha-Pouliquen, JFM ['09]
    - Iverson-George, Proc. Roy. Soc. London A ['14]

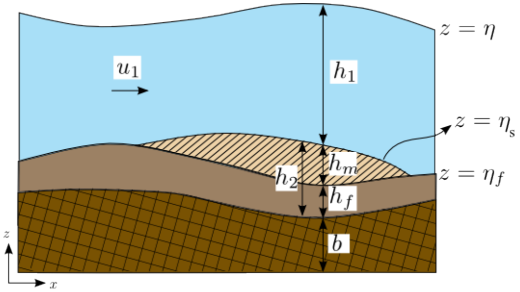
Bouchut - Fernandez Nieto - Mangeney - Narbona Reina, JFM ['16]

Sediment transport : Derivation of SVE type models

- Focus on bedload transport
- Derivation from Navier-Stokes equations
- Fluids with different properties and different time scales
- Thin layer asymptotics

⇝ Generalized Saint-Venant – Exner type models

⇝ Energy inequality



Sediment transport : Derivation of SVE type models

- Fluid layer : Newtonian

$$\sigma_f = 2\mu_f D(u_f)$$

- Sediment layer : Drucker-Prager type

$$\sigma_s = 2\mu_s(p_s, D(u_s))D(u_s)$$

- Fluid-Sediment interface : Navier friction

$$(\sigma_f n_{sf}) \cdot \tau_{sf} = (\sigma_s n_{sf}) \cdot \tau_{sf} = C|u_f - u_s|^m(u_f - u_s)$$

- Sediment-Bed interface : Coulomb friction

$$(\sigma_s n_{sb}) \cdot \tau_{sb} = -(\mathrm{sgn}(u_s) \tan\delta \ ((\sigma_f - \sigma_s)n_{sb}) \cdot n_{sb})$$

Fernandez Nieto - Morales de Luna - Narbona Reina - Zabsonre, M2AN ['16]

Sediment transport : Derivation of SVE type models

- Fluid layer : Shallow Water asymptotics
- Sediment layer : Lubrication (Reynolds) asymptotics
- Two time scales

$$T_f = \epsilon^2 T_s$$

- Saint-Venant – Exner model

$$\partial_t h_f + \nabla \cdot q_f = 0,$$

$$\partial_t q_f + \nabla \cdot (h_f (u_f \otimes u_f)) + \frac{1}{2} g \nabla h_f^2$$

$$+ g h_f \nabla (b + h_s) + \frac{g h_{s,u}}{r} \mathcal{P} = 0,$$

$$\partial_t h_s + \nabla \cdot (h_{s,u}\, u_s(u_f, \mathcal{P})) = 0,$$

$$\partial_t h_{s,b} = -E + D$$

with

$$\mathcal{P} = \nabla (r h_1 + h_2 + b) + (1 - r)\mathrm{sgn}(u_s) \tan \delta$$

Sediment transport : Derivation of SVE type models

- Fluid layer : Shallow Water asymptotics
- Sediment layer : Lubrication (Reynolds) asymptotics
- Two time scales

$$T_f = \epsilon^2 T_s$$

- MPM type SVE model $(E = D)$

$$\partial_t h_f + \nabla \cdot q_f = 0,$$

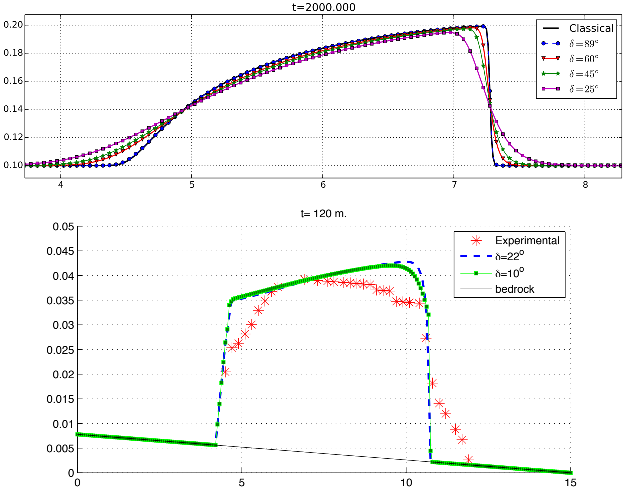
$$\partial_t q_f + \nabla \cdot (h_f(u_f \otimes u_f)) + \frac{1}{2} g \nabla h_f^2 + g h_f \nabla (b + h_s) + \frac{g h_{s,u}}{r} \mathcal{P} = 0,$$

$$\partial_t h_s + \nabla \cdot \left(\frac{k_1 \mathrm{sgn}(\tau_{\mathrm{eff}})}{1 - \phi} (\theta_{\mathrm{eff}} - \theta_c)_+^{3/2} \sqrt{(1/r - 1) g d_s} \right) = 0$$

with

$$\frac{\tau_{\mathrm{eff}}}{\rho_f} = \frac{\vartheta\, d_s}{h_s} \frac{\tau}{\rho_f} - \frac{g d_s \vartheta}{r} \nabla (r h_f + h_s + b). \qquad \theta_{\mathrm{eff}} = \frac{|\tau_{\mathrm{eff}}| / \rho_f}{(1/r - 1) g d_s}$$

Sediment transport : Derivation of SVE type models



Fernandez Nieto - Morales de Luna - Narbona Reina - Zabsonre, M2AN ['16]

Sediment transport : Derivation of SVE type models

- Fluid layer : Newtonian, Small viscosity
- Sediment layer : Newtonian, (Very) Large viscosity
- Fluid-Sediment interface : Navier friction (possible threshold)
- Sediment-Bed interface : Navier friction

⇝ Modified Shallow Water asymptotics

$$\begin{aligned} \partial_t h_f + \nabla \cdot (h_f u_f) &= 0, \\ \partial_t (h_f u_f) + \nabla \cdot (h_f u_f \otimes u_f + \frac{g h_f^2}{2} \mathrm{Id}) &= -g h_f \nabla (h_s + b) - \kappa_i u_f. \\ \partial_\tau h_s + \nabla \cdot (h_s u_s) &= 0 \end{aligned}$$

where u_s is solution of

$$\mathcal{L}(u_s) = -g h_s \nabla (h_s + r h_f + b) + \kappa_i u_f.$$

with

$$\mathcal{L}(\phi) = \kappa_b \phi$$

Audusse - Boittin - Parisot ['17]

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where u_s is solution of

$$\mathcal{L}(u_s) = -g h_s \nabla (h_s + r h_f + b) + \kappa_i u_f.$$

with $\mathcal{L}$ a non-local operator

$$\mathcal{L}(\phi) = \kappa_b \phi - \alpha \nabla \cdot (\mu_s h_s \mathrm{D}_x \phi)$$

Audusse - Boittin - Parisot ['17]

Conclusions

- Shallow Water type models
- Robust numerical methods (mainly Finite Volumes)

- Asymptotics expansion of 3D models
- Interaction with other communities

$\rightsquigarrow$ Layerwise approach for Navier-Stokes

$\rightsquigarrow$ Complex rheology

$\rightsquigarrow$ Multiphase flows

$\rightsquigarrow$ Sediment transport

- EGRIN 2 (2017-2021)

http ://gdr-egrin.math.cnrs.fr/