Derivation of a bedload transport model with viscous effects

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May 30, 2017

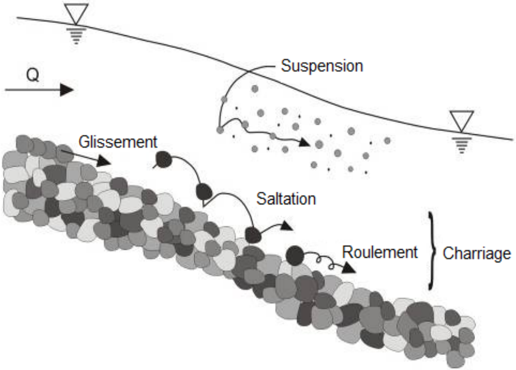
Why should we simulate sediment transport?

- Predict the evolution of the river topography
- Estimate sediment accumulation at the bottom of dams, in harbours...
- Estimate the stability of structures such as canals and bridges with scour

Sediment accumulation

Scouring

Different modes of sediment transport



Source: *http://theses.univ-lyon2.fr/documents/getpart.php?id=lyon2.2008.pintomartins_d&part=154405*

Classical bed load transport model: Exner equation

Clear water → Saint-Venant equations

Sediment layer → Exner equation
Based on mass conservation:

$$\partial_t h_2 + \nabla \cdot q_2 = 0$$

- h_2: thickness of sediment layer
- Solid discharge q_2 given by empirical relationships

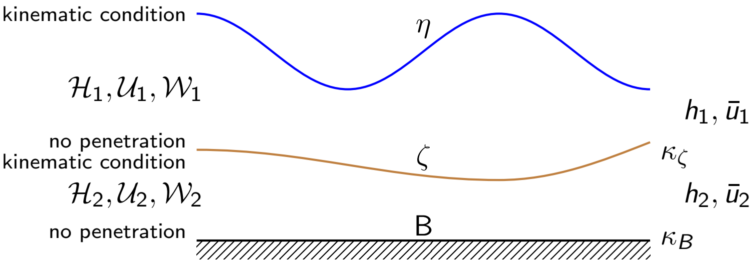
  Einstein (1942), Meyer-Peter and Müller (1948), Nielsen (1992)
- q_2 depends on the water depth h_1 and velocity u_1:
  no intrinsic mechanism in the sediment

Goals: derive and simulate a new bedload transport model

- Formal derivation of a model, from the Navier-Stokes equations
- Coupled model with an energy equation

E.D. Fernández-Nieto et al. "Formal deduction of the Saint-Venant-Exner model including arbitrarily sloping sediment beds and associated energy". In: *Mathematical Modelling and Numerical Analysis* (2016)

- Try to include the sediment rheology in the model



Navier-Stokes equations

→

- thin layer approximation
- high viscosity, high friction in the sediment layer

Saint-Venant equations, sediment transport model

Inviscid model with high friction

Assume the following scaling:

$$\begin{array}{lll} \frac{H}{L} = \varepsilon & W = \varepsilon U, & \\ \frac{1}{Fr} = 1, & Re_1 = \frac{1}{\varepsilon}, & Re_2 = 1, \\ K_\zeta = \varepsilon, & K_B = 1. & \end{array}$$

Assume that the space variations of the μ_k are of the order ε^2. Then, for ε small enough, the system

$$\begin{cases} \partial_t h_1 + \nabla_x \cdot (h_1 \bar{u}_1) = 0, \\ \partial_t (h_1 \bar{u}_1) + \nabla_x \cdot (h_1 \bar{u}_1 \otimes \bar{u}_1 + \frac{h_1^2}{2Fr^2} \mathrm{Id}) = -\frac{h_1}{Fr^2} \nabla_x (h_2 + B) - \frac{1}{r} \kappa_\zeta (\bar{u}_1 - \varepsilon \tilde{u}_2), \end{cases}$$

$$\begin{cases} \partial_t h_2 + \varepsilon \nabla_x \cdot (h_2 \tilde{u}_2) = 0, \\ \kappa_B(\tilde{u}_2) = -\frac{h_2}{Fr^2} \nabla_x (h_2 + r h_1 + B) + \kappa_\zeta (\bar{u}_1 - \varepsilon \tilde{u}_2), \end{cases}$$

with $\kappa_B(\tilde{u}_2) = \tilde{\kappa}_B \tilde{u}_2$ is derived from the bilayer Navier-Stokes system with the following modeling errors:

$$\begin{array}{ll} |\mathcal{H}_1 - h_1| = O(\varepsilon), & |\mathcal{H}_2 - h_2| = O(\varepsilon^2), \\ |\mathcal{U}_1 - \bar{u}_1| = O(\varepsilon), & |\mathcal{U}_2 - \varepsilon \tilde{u}_2| = O(\varepsilon^2). \end{array}$$

Proof

Momentum eq. and boundary conditions:

$$\left.\begin{array}{ll} \partial_z(\mu_2 \partial_z \mathcal{U}_2) = O(\varepsilon^2), & \\ \mu_2 \partial_z \mathcal{U}_2 = O(\varepsilon^2), & \text{at } z = \zeta \\ \mu_2 \partial_z \mathcal{U}_2 = \varepsilon \kappa_B \mathcal{U}_2 + O(\varepsilon^2), & \text{at } z = B \end{array}\right\} \Rightarrow \mu_2 \partial_z \mathcal{U}_2|_B = O(\varepsilon^2)$$

Imposes that $\mathcal{U}_2 = \bar{U}_2 + O(\varepsilon^2) = \varepsilon \tilde{U}_2 + O(\varepsilon^2)$.

Fine, but similar to an Exner model, and **no rheology...**

Viscous model

Assume the following scaling:

$$\begin{array}{lll} H/L = \varepsilon & W = \varepsilon U, & \\ \frac{1}{Fr} = 1, & Re_1 = \frac{1}{\varepsilon}, & Re_2 = \varepsilon, \\ K_\zeta = \varepsilon, & K_B = 1. & \end{array}$$

Assume that the space variations of the μ_k are of the order ε^2. Then, for ε small enough, the system

$$\begin{cases} \partial_t h_1 + \nabla_x \cdot (h_1 \bar{u}_1) = 0, \\ \partial_t (h_1 \bar{u}_1) + \nabla_x \cdot (h_1 \bar{u}_1 \otimes \bar{u}_1 + \frac{h_1^2}{2Fr^2} \mathrm{Id}) = -\frac{h_1}{Fr^2} \nabla_x (h_2 + B) - \frac{1}{r} \kappa_\zeta (\bar{u}_1 - \varepsilon \tilde{u}_2), \end{cases}$$

$$\begin{cases} \partial_t h_2 + \varepsilon \nabla_x \cdot (h_2 \tilde{u}_2) = 0, \\ \kappa_B(\tilde{u}_2) = -\frac{h_2}{Fr^2} \nabla_x (h_2 + r h_1 + B) + \kappa_\zeta (\bar{u}_1 - \varepsilon \tilde{u}_2), \end{cases}$$

with $\kappa_B(\tilde{u}_2) = \tilde{\kappa}_B \tilde{u}_2 - \nabla_x \cdot (\mu_2 h_2 D_x \tilde{u}_2)$ is derived from the bilayer Navier-Stokes system with the following modeling errors:

$$\begin{array}{ll} |\mathcal{H}_1 - h_1| = O(\varepsilon), & |\mathcal{H}_2 - h_2| = O(\varepsilon^2), \\ |\mathcal{U}_1 - \bar{u}_1| = O(\varepsilon), & |\mathcal{U}_2 - \varepsilon \tilde{u}_2| = O(\varepsilon^2). \end{array}$$

Proof

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Vertically integrated horizontal momentum equation:

$$\begin{aligned} \partial_t(\mathcal{H}_2\bar{\mathcal{U}}_2) \quad & +\nabla_x\cdot(\mathcal{H}_2\bar{\mathcal{U}}_2\otimes\bar{\mathcal{U}}_2) + \frac{\mathcal{H}_2}{Fr^2}\nabla_x(rh_1+\mathcal{H}_2+B) \\ & = -\frac{\tilde{\kappa}_B\bar{\mathcal{U}}_2}{\varepsilon} + \frac{1}{\varepsilon}\nabla_x\cdot(\mu_2 h_2 \mathrm{D}_x\bar{\mathcal{U}}_2) - \kappa_\zeta(\bar{\mathcal{U}}_2 - \bar{u}_1) + O(\varepsilon), \end{aligned}$$

Main order terms:

$$\tilde{\kappa}_B\bar{\mathcal{U}}_2 - \nabla_x\cdot(\mu_2 h_2 \mathrm{D}_x\bar{\mathcal{U}}_2) = O(\varepsilon).$$

Imposes that $\bar{\mathcal{U}}_2 = O(\varepsilon) = \varepsilon\tilde{\mathcal{U}}_2 + O(\varepsilon^2)$.

Threshold for incipient motion

Classical laws for sediment transport used in hydraulic engineering have a **threshold** for incipient motion (eg. Meyer-Peter and Müller).

Critical shear stress: τ_c
Effective shear stress: $\tau_{eff} = \kappa_\zeta \bar{u}_1 - \frac{h_2}{Fr^2} \nabla_x (rh_1 + h_2 + B)$
Velocity equation: $\kappa_B(\tilde{u}_2) = \tau_{eff}$
Take $\kappa_B(\cdot)$ such that $\kappa_B(\bar{u}_2) = (\tilde{\kappa}||\bar{u}_2||^\alpha + \tau_c)\frac{\tau_{eff}}{||\tau_{eff}||} - \nabla_x \cdot (\mu_2 h_2 \nabla_x \bar{u}_2)$, with

$$\tilde{\kappa} = \begin{cases} \tilde{\kappa} \text{ if } ||\tau_{eff}|| \geq \tau_c \\ +\infty \text{ if } ||\tau_{eff}|| < \tau_c \end{cases}$$

Model analysis

Dissipative energy balance for the bilayer model

For smooth enough solutions:

$$\partial_t(\mathcal{K}_1 + \mathcal{E}) + \nabla_x \cdot (\mathcal{K}_1 u_1 + h_1 \phi_1 \bar{u}_1 + \frac{\varepsilon}{r} h_2 \phi_2 \tilde{u}_2) = -\frac{\varepsilon}{r} \tilde{u}_2 \cdot \kappa_B(\tilde{u}_2) - \frac{\kappa_\zeta}{r} |\bar{u}_1 - \varepsilon \tilde{u}_2|^2,$$

with $\mathcal{K}_1 = \dfrac{1}{2} h_1 |\bar{u}_1|^2$: kinematic energy of the water

$\mathcal{E} = \frac{1}{Fr^2}\left(h_1\left(\frac{h_1}{2} + h_2 + B\right) + \frac{h2}{r}\left(\frac{h_2}{2} + B\right)\right)$: potential energy

$\phi_1 = \frac{1}{Fr^2}(h_2 + h_1 + B), \phi_2 = \frac{1}{rFr^2}(h_2 + rh_1 + B)$: potentials

Sediment layer only, without forcing

- Positivity
- Maximum principle for smooth solutions

A first idea for the numerical scheme

Simplified model:

Sediment layer alone
No viscosity $\Rightarrow$ nonlinear heat equation
No topography

$$\partial_t h_2 - \varepsilon \frac{\kappa_B}{Fr^2} \nabla_x \cdot (h_2^2 \nabla_x h_2) = 0.$$

- Explicit scheme: parabolic CFL condition $\Delta t \leq C(\Delta x)^2$
- Try an implicit scheme

3 different schemes

Implicit (linearized) finite volume schemes, staggered grid

$$\begin{cases} h_i^{n+1} & = h_i^n - \frac{\Delta t}{\Delta x}\left(h_{i+1/2}^n u_{i+1/2}^{n+1} - h_{i-1/2}^n u_{i-1/2}^{n+1}\right) \\ u_{i+1/2}^{n+1/2} - & \frac{\nu}{(\Delta x)^2}\left(h_{i+1}^n\left(u_{i+3/2}^{n+1} - u_{i+1/2}^{n+1}\right) - h_i^n\left(u_{i+1/2}^{n+1} - u_{i-1/2}^{n+1}\right)\right) \\ & = -g h_{i+1/2}^n \frac{h_{i+1}^{n+1} - h_i^{n+1}}{\Delta x}, \end{cases}$$

This scheme **dissipates** the discrete energy.

- Centered scheme

Take $h_{i+1/2}^n = \frac{h_i^n + h_{i+1}^n}{2}$

- Upwind with respect to ∇h

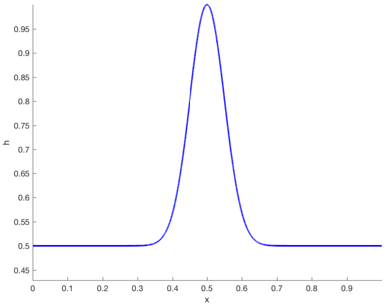
Take $h_{i+1/2}^n = \max(h_i^n, h_{i+1}^n)$

- Upwind with respect to u

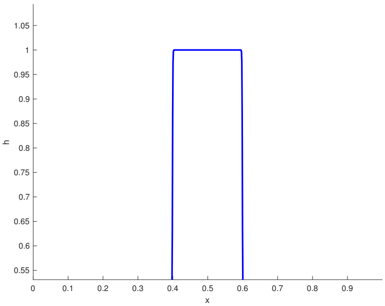
Take $h_{i+1/2}^n = \begin{cases} h_i^n \text{ if } u_{i+1/2}^{n+1} \geq 0 \\ h_{i+1}^n \text{ if } u_{i+1/2}^{n+1} < 0 \end{cases}$. A fixed point is needed.

Comparison of the schemes

Regular



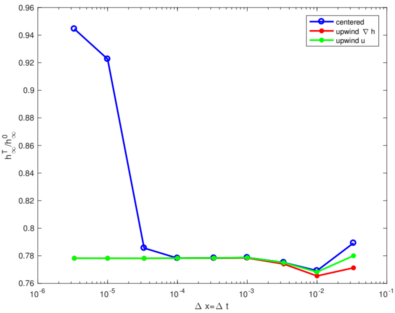
Irregular



Initial conditions

Comparison of the schemes

Regular



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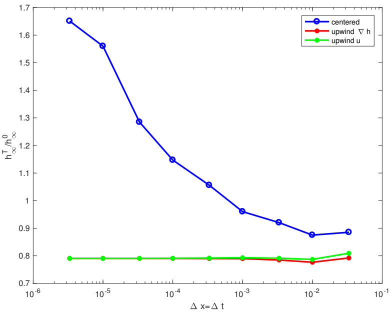
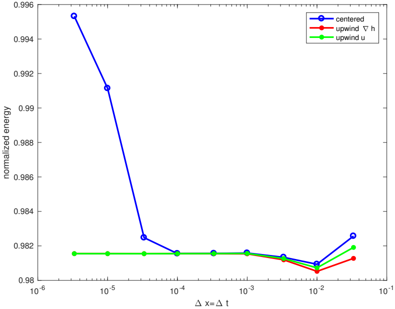


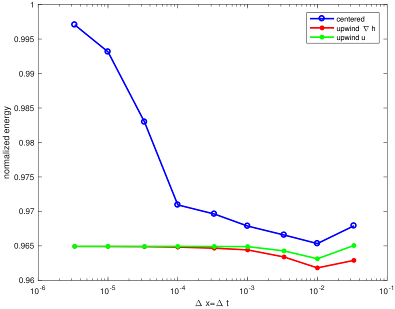
Infinity norm at final time

Comparison of the schemes

Regular



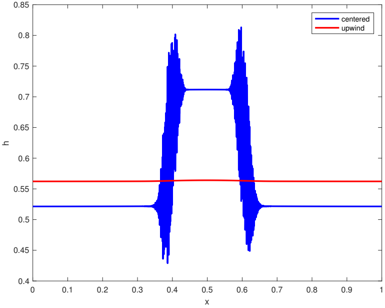
Irregular



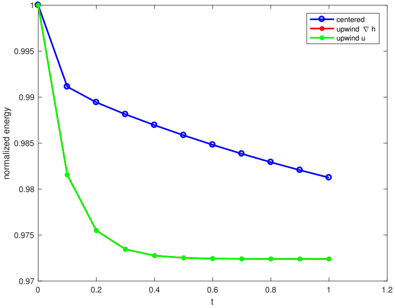
Energy at final time

Problems with the centered scheme

Energy dissipation, but oscillations!



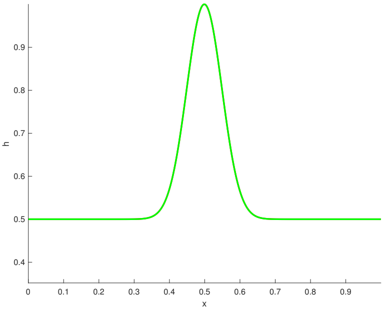
Solution at final time, starting from smooth initial condition



Energy dissipation for the three schemes

Simulation of the forced model

No topography: $B(x) = 0$. Constant free surface: $\eta = rh_1 + h_2$ =constant
The two upwind schemes behave differently



Water velocity $u_1 = 10$, density ratio $r = 0.6$

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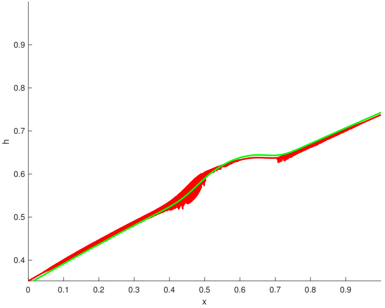
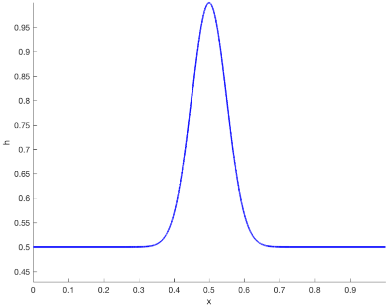
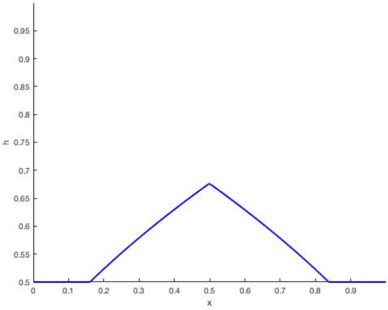


Water velocity $u_1 = 10$, density ratio $r = 0.6$

Simulations with a threshold in the friction coefficient



Simulations with a threshold in the friction coefficient



Non-flat stationary states

Why are the simulations unstable?

An idea: antidiffusion fluxes

Simplified version of the equations for the sediment layer in 1D:

$$\partial_t h_2 + \varepsilon \partial_x h_2 \tilde{u}_2 = 0,$$
$$\tilde{u}_2 - \partial_{xx}^2 \tilde{u}_2 = -\partial_x h_2$$

New variable:

$$D = -\frac{h_2 \tilde{u}_2 \partial_x h_2}{|\partial_x h_2|^2}$$

Continuity equation:

$$\partial_t h_2 - \varepsilon \partial_x (D \partial_x h_2) = 0,$$

well-posed if and only if $D > 0$. This may not always be the case everywhere in the domain...

Conclusions

A new model for sediment transport with viscosity

- Formal derivation
- Preliminary analysis
- Comparison of three schemes on a staggered grid

Future work

- Comparison with co-located schemes
- Find the cause(s) of the instabilities in the simulations (antidiffusion?)
- Simulate the coupled system (water+sediment)
  - staggered grid [Gunawan *et al.* '14]
  - co-located grid
- More physical rheologies (Bingham?)

Why doesn't the order of approximation in the water layer ruin the approximations in the sediment layer?

$$\begin{aligned}
\partial_t(\mathcal{H}_2\bar{\mathcal{U}}_2) \qquad & +\nabla_x\cdot(h\bar{\mathcal{U}}_2\otimes\bar{\mathcal{U}}_2)+\tfrac{\mathcal{H}_2}{Fr^2}\nabla_x(r\mathcal{H}_1+\mathcal{H}_2+B)\\
&=-\tfrac{\tilde{\kappa}_B\bar{\mathcal{U}}_2}{\varepsilon}+\tfrac{1}{\varepsilon}\nabla_x\cdot(\mu_2 h_2\mathrm{D}_x\bar{\mathcal{U}}_2)-\kappa_\zeta(\bar{\mathcal{U}}_2-\bar{u}_1)+O(\varepsilon^2),\\
\partial_t(\mathcal{H}_2\bar{\mathcal{U}}_2) \qquad & +\nabla_x\cdot(h\bar{\mathcal{U}}_2\otimes\bar{\mathcal{U}}_2)+\tfrac{\mathcal{H}_2}{Fr^2}\nabla_x(rh_1+\mathcal{H}_2+B)\\
&=-\tfrac{\tilde{\kappa}_B\bar{\mathcal{U}}_2}{\varepsilon}+\tfrac{1}{\varepsilon}\nabla_x\cdot(\mu_2 h_2\mathrm{D}_x\bar{\mathcal{U}}_2)-\kappa_\zeta(\bar{\mathcal{U}}_2-\bar{u}_1)+O(\varepsilon),
\end{aligned}$$

And then

$$\tilde{\kappa}_B\bar{\mathcal{U}}_2-\nabla_x\cdot(\mu_2 h_2\mathrm{D}_x\bar{\mathcal{U}}_2)=O(\varepsilon).$$

Imposes $\bar{\mathcal{U}}_2=O(\varepsilon)=\varepsilon\tilde{\mathcal{U}}_2+O(\varepsilon^2)$. Then:

$$\partial_t\mathcal{H}_2+\varepsilon\nabla_x\cdot(\mathcal{H}_2\tilde{\mathcal{U}}_2)=0,$$
$$\partial_t\mathcal{H}_2+\varepsilon\nabla_x\cdot(\mathcal{H}_2\tilde{u}_2)=O(\varepsilon^2)$$

Then $h_2=\mathcal{H}_2+O(\varepsilon^2)$.