Multi-dimensional shear shallow water flows

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Shear shallow water equations (Teshukov (2007), S. Gavrilyuk, G. Richard (2013))

$$\begin{cases} \dfrac{\partial h}{\partial t} + div(h\mathbf{u}) = 0 \\ \\ \dfrac{\partial h\mathbf{u}}{\partial t} + div(h\mathbf{u} \otimes \mathbf{u} + g\frac{h^2}{2}\mathbf{I} + h\mathbf{P}) = \mathbf{F}, \\ \dfrac{D\mathbf{P}}{Dt} + \dfrac{\partial \mathbf{u}}{\partial \mathbf{x}}\mathbf{P} + \mathbf{P}\left(\dfrac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T = \mathbf{D} \end{cases}$$

$\dfrac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla_2$
$\mathbf{u} = (u, v)^T$ the average velocity :
$\mathbf{u}(t, \mathbf{x}) = \frac{1}{h}\int_0^h \tilde{\mathbf{u}}(t, x, y, z)dz$
$\mathbf{x} = (x, y)^T$
h : the fluid layer thickness
$\mathbf{F} = -gh\nabla\mathbf{b} - C_f\mathbf{u}|\mathbf{u}|$
$\mathbf{P} = \frac{1}{h}\int_0^h (\tilde{\mathbf{u}} - \mathbf{u}) \otimes (\tilde{\mathbf{u}} - \mathbf{u})\, dz.$

The equations are hyperbolic if **P** is positive definite. This system is not conservative in $2D$ case : we have only 5 conservation laws for 6 variables. Equations admit the energy conservation law :

$$\frac{\partial}{\partial t}\left(h\left(\frac{1}{2}|\mathbf{u}|^2 + e\right)\right) + \mathrm{div}\left(h\mathbf{u}\left(\frac{1}{2}|\mathbf{u}|^2 + e\right) + \left(\frac{gh^2}{2}\mathbf{I} + h\mathbf{P}\right)\mathbf{u}\right) = \mathbf{F}\cdot\mathbf{u} - Q,$$

where

$$e = \frac{1}{2}\mathrm{tr}\mathbf{P} + \frac{1}{2}gh,\ Q > 0,$$

and the conservation of 'entropy' :

$$\frac{\partial(h\Psi)}{\partial t} + \mathrm{div}\,(h\mathbf{u}\Psi) = S(h, \mathbf{u}, \mathbf{P}),\ \Psi = \frac{Det\,(\mathbf{P})}{h^2}.$$

Splitting in Cartesian coordinates

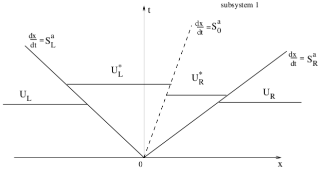
The subsystem in x direction :

$$
\begin{cases}
h_t + (hu)_x = 0 \\
(hu)_t + \left(hu^2 + \dfrac{gh^2}{2} + hP_{11} \right)_x = 0 \\
(hv)_t + (huv + hP_{12})_x = 0, \\
P_{11t} + uP_{11x} + 2P_{11} u_x = 0, \\
P_{12t} + (uP_{12})_x + P_{11} v_x = 0, \\
P_{22t} + uP_{22x} + 2P_{12} v_x = 0.
\end{cases}
$$

System is hyperbolic and admits three types of waves : a contact discontinuity u, surface gravity waves $u \pm a$, where $a^2 = gh + 3P_{11}$, and shear waves $u \pm b$, $b^2 = P_{11}$.

a - waves

$$
\begin{cases}
h_t + (uh)_x = 0, & (a)\\
(hu)_t + \left(hu^2 + g\frac{h^2}{2} + hP_{11}\right)_x = 0, & (b)\\
(hv)_t + (huv)_x = 0, & (c)\\
(hP_{11})_t + (huP_{11})_x + 2hP_{11}u_x = 0, & (d)\\
P_{12t} + (uP_{12})_x = 0, & (e)\\
(hP_{22})_t + (huP_{22})_x = 0. & (f)
\end{cases}
$$



The subsystem admits the energy conservation law

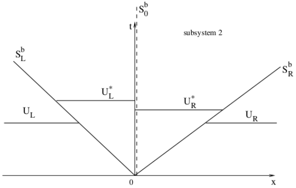
$$
\left(h\left(\frac{u^2+v^2}{2} + \frac{gh}{2} + \frac{P_{11}+P_{22}}{2}\right)\right)_t
$$

$$
+ \left(hu\left(\frac{u^2+v^2}{2} + \frac{gh}{2} + \frac{P_{11}+P_{22}}{2}\right) + \frac{gh^2}{2}u + hP_{11}u\right)_x = 0.
$$

The equation (d) is not conservative and the value of P_{11} will be corrected using the energy conservation law.

b - waves

$$\begin{cases} h_t = 0, & (a) \\ (hu)_t = 0, & (b) \\ (hv)_t + (hP_{12})_x = 0, & (c) \\ (hP_{11})_t = 0, & (d) \\ P_{12t} + P_{11} v_x = 0, & (e) \\ (hP_{22})_t + 2hP_{12} v_x = 0. & (f) \end{cases}$$



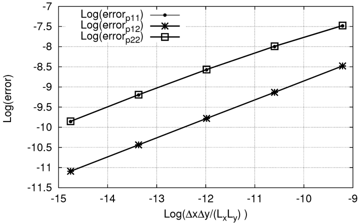
It also admits the energy conservation law :

$$\left(h \left(\frac{u^2 + v^2}{2} + \frac{P_{11} + P_{22}}{2} + \frac{gh}{2} \right) \right)_t + (hP_{12} v)_x = 0. \qquad (1)$$

The energy conservation law was used for update the value of P_{22}.

Analytical solution

$$\begin{cases} h = \dfrac{h_0}{1+\beta^2 t^2}, \\ \mathbf{U} = \dfrac{\beta}{1+\beta^2 t^2}\begin{pmatrix} \beta t x + y \\ -x + \beta t y \end{pmatrix}, \\ \mathbf{P} = \dfrac{1}{(1+\beta^2 t^2)^2}\begin{pmatrix} \lambda + \gamma\beta^2 t^2, & (\lambda-\gamma)\beta t \\ (\lambda-\gamma)\beta t, & \gamma + \lambda\beta^2 t^2 \end{pmatrix}, \end{cases}$$



where $h_0 > 0$, β, $\lambda > 0$, $\gamma > 0$ are constant. $h_0 = 1\ m$, $\lambda = 0.1\ m^2/s^2$, $\gamma = 0.01\ m^2/s^2$, $\beta = 10^{-3}\ s^{-1}$.

$$error(P_{11}) = max_{x,y}\left(\frac{|P_{11numerical} - P_{11analytical}|}{\lambda}\right),$$

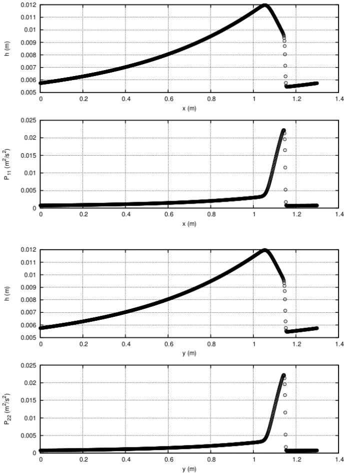
$$error(P_{12}) = max_{x,y}\left(\frac{|P_{12numerical} - P_{12analytical}|}{\lambda+\gamma}\right),$$

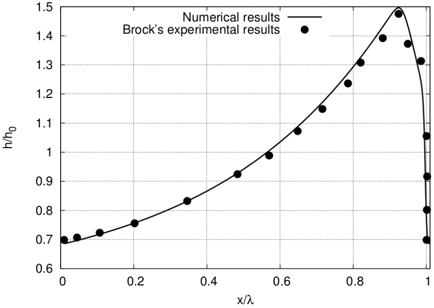
$$error(P_{22}) = max_{x,y}\left(\frac{|P_{22numerical} - P_{22analytical}|}{\gamma}\right).$$

Application to roll waves

Numerical results for 2D full non-conservative model in periodic box

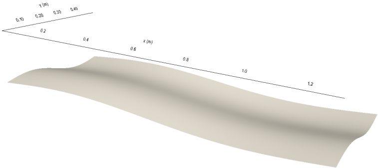
TEST	h_0 $[m]$	θ $[rad]$	C_r	C	φ $[s^{-2}]$	λ $[m]$	ω $[rad/s]$	T $[s]$	Fr_{g0}	a
CASE 1	0.00798	0.05011	0.00035	0.0036	22.76	≈ 1.3	6.73	0.933	3.63	0.05





Formation of a transverse structure of the jump toe perimeter

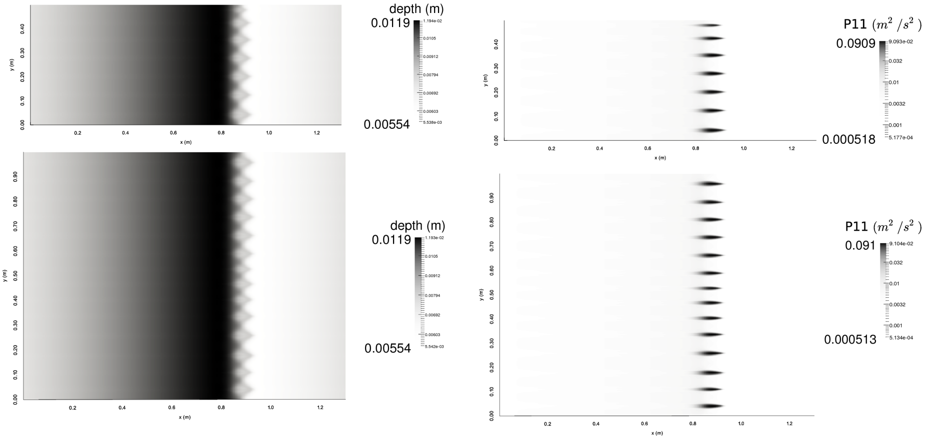
$L_x = 1.3m,\ L_y = 0.5m$ with 600×600 grid cells.



IC : $h = h_0 \left(1 + a \sin\left(\frac{2\pi x}{L_x}\right) + \sin\left(\frac{2\pi y}{L_y}\right)\right),\ u = \sqrt{\frac{g h_0 \tan\theta}{C_f}},\ v = 0,\ P_{11} = P_{22} = \frac{\varphi h(x,y,0)^2}{2},\ P_{12} = 0.$

BC : In the direction of wave propagation we use periodic conditions, and in the transverse direction we use the rigid wall conditions.

Transverse front structure does not depend on the initial perturbations and on the channel width !



Time = 39 s

Full model in polar coordinates without RHS

$$
\begin{cases}
\dfrac{\partial(hr)}{\partial t} + \dfrac{\partial(rhU_r)}{\partial r} + \dfrac{\partial(hU_\theta)}{\partial \theta} = 0, \\
\dfrac{\partial(rhU_r)}{\partial t} + \dfrac{\partial}{\partial r}\left(r\left(hU_r^2 + \dfrac{gh^2}{2} + hP_{rr}\right)\right) + \dfrac{\partial}{\partial \theta}(hU_rU_\theta + hP_{r\theta}) = h(U_\theta^2 + P_{\theta\theta}) + \frac{gh^2}{2}, \\
\dfrac{\partial\left(rhU_\theta\right)}{\partial t} + \dfrac{\partial}{\partial r}\left(rh\left(U_rU_\theta + P_{r\theta}\right)\right) + \dfrac{\partial}{\partial \theta}\left(hU_\theta^2 + \dfrac{gh^2}{2} + hP_{\theta\theta}\right) = -h\left(U_rU_\theta + P_{r\theta}\right), \\
\dfrac{\partial P_{rr}}{\partial t} + U_r\dfrac{\partial P_{rr}}{\partial r} + \dfrac{U_\theta}{r}\left(\dfrac{\partial P_{rr}}{\partial \theta} - 4P_{r\theta}\right) + 2\left(\dfrac{\partial U_r}{\partial r}P_{rr} + \dfrac{P_{r\theta}}{r}\dfrac{\partial U_r}{\partial \theta}\right) = 0, \\
\dfrac{\partial P_{r\theta}}{\partial t} + U_r\dfrac{\partial P_{r\theta}}{\partial r} + \dfrac{U_\theta}{r}\left(\dfrac{\partial P_{r\theta}}{\partial \theta} + P_{rr} - 2P_{\theta\theta}\right) + \dfrac{P_{\theta\theta}}{r}\dfrac{\partial U_r}{\partial \theta} + P_{rr}\dfrac{\partial U_\theta}{\partial r} + \dfrac{P_{r\theta}}{r}\left(\dfrac{\partial U_\theta}{\partial \theta} + \dfrac{\partial (rU_r)}{\partial r}\right) = 0, \\
\dfrac{\partial P_{\theta\theta}}{\partial t} + U_r\dfrac{\partial P_{\theta\theta}}{\partial r} + \dfrac{U_\theta}{r}\left(\dfrac{\partial P_{\theta\theta}}{\partial \theta} + 2P_{r\theta}\right) + 2\left(\dfrac{\partial U_\theta}{\partial r}P_{r\theta} + \dfrac{P_{\theta\theta}}{r}\left(\dfrac{\partial U_\theta}{\partial \theta} + U_r\right)\right) = 0, \\
\dfrac{\partial}{\partial t}\left(hr\left(\dfrac{1}{2}|\mathbf{U}|^2 + E\right)\right) + \dfrac{\partial}{\partial r}\left(r\left(hU_r\left(\dfrac{1}{2}|\mathbf{U}|^2 + E\right) + \dfrac{gh^2}{2}U_r + h(P_{rr}U_r + P_{r\theta}U_\theta)\right)\right) + \\
+\dfrac{\partial}{\partial \theta}\left(hU_\theta\left(\dfrac{1}{2}|\mathbf{U}|^2 + E\right) + \dfrac{gh^2}{2}U_\theta + h(P_{r\theta}U_r + P_{\theta\theta}U_\theta)\right) = 0.
\end{cases}
$$

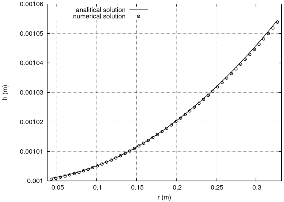
where

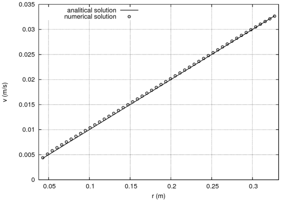
$E = (gh + P_{rr} + P_{\theta\theta})/2.$

Water glass on a record player

The following set of equalities is exact solution of our system and correspond to the profile of the water in the glass turning on a record player :

$$\begin{cases} h(r,\theta,t) = h_0 + \dfrac{\omega^2 r^2}{2g}, \\ \qquad U_r(r,\theta,t) = 0, \qquad\qquad U_\theta(r,\theta,t) = \omega r, \\ \qquad \mathbf{P}(r,\theta,t) = \mathbf{0}. \end{cases}$$





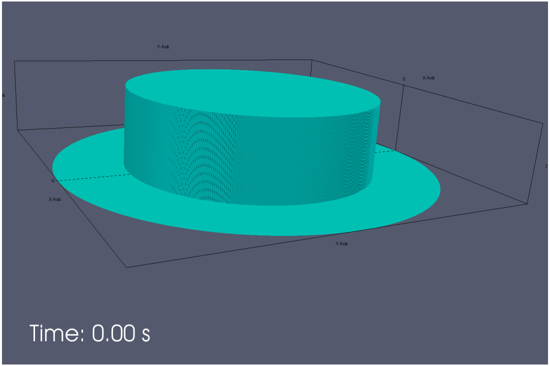
A comparison between analitical and numerical solution at time instant $t = 50\ s$ with 50 points grid. HLLC Riemann solver was used. $CFL = 0.9$, $\omega = 0.1$.

Radial Dam-Break

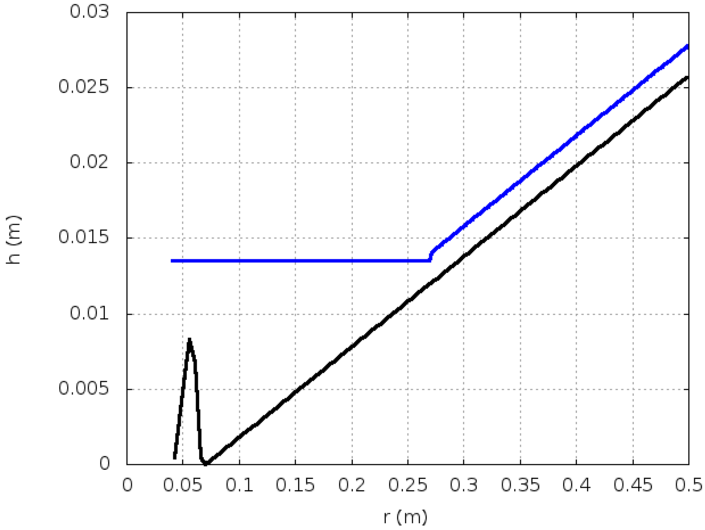
Computational domain here $0.5\ m < r < 1\ m$. The initial water height is

$$h(r,0) = \begin{cases} 0.02m, & if \quad r \leq 0.5L_r, \\ 0.01m, & if \quad r > 0.5L_r \end{cases}$$

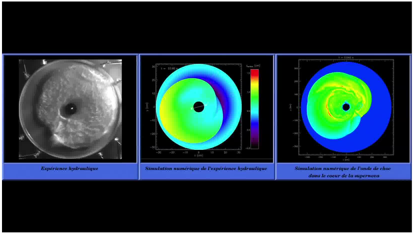
At the inward boundary and outward boundary we apply wall type boundaries.

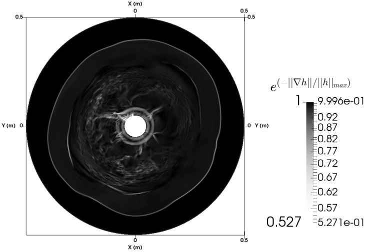


1D hydraulic jump



Radial hydraulic jump





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Conclusion

1. A new mathematical model is proposed to study multi-dimensional shear flows
2. The model predicts, in particular, the formation of transverse structures observed in filed experiments
3. The model also predicts the bifurcation of radial oscillation of the hydraulic jump into the its rotation.

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