An adaptive numerical scheme for solving incompressible free-surface flows

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Free-surface incompressible flows

Incompressible Navier–Stokes equations :

$$\begin{cases} \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f} \\ div\ \mathbf{u} = 0 \\ \mathbf{u}(0,x) = \mathbf{u}_0(x). \end{cases} \tag{NS}$$

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Difficulties

1. Mathematical complexity of the Navier–Stokes equations
2. Time evolution of the fluid domain

Approaches

1. Study of shallow water approximations of incompressible Navier–Stokes/Euler (Advantages : simplicity, dimension reduction)
2. Study directly incompressible Navier–Stokes/Euler (Advantages : precision, general regime)

Plan

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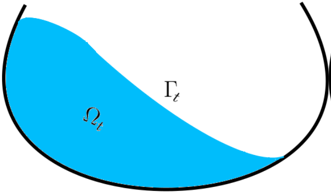
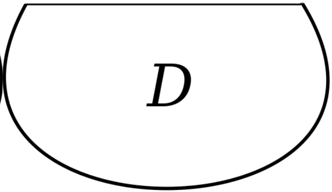
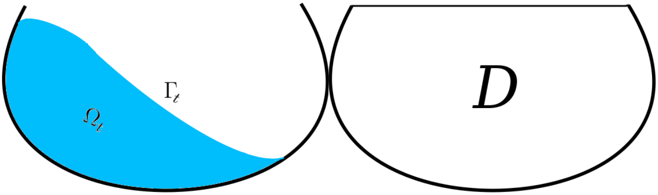
Mathematical model

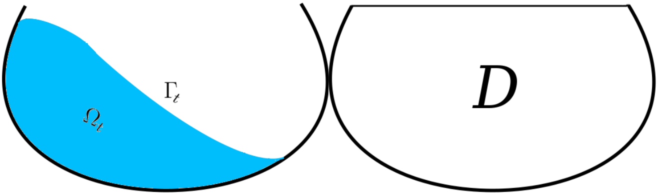






Fluid equations :

$$\begin{cases} \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)u\right) - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f} \quad \text{in} \quad \Omega_t, \\ div\ \mathbf{u} = 0 \quad \text{in} \quad \Omega_t, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) \quad \text{in} \quad \Omega_0. \end{cases} \tag{1}$$



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Boundary conditions :

$$\begin{cases} \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega_t \setminus \Gamma_t, \\ \left[\alpha\mathbf{u} + \mu\left(\nabla\mathbf{u} + {}^t\nabla\mathbf{u}\right)\mathbf{n}\right]_{tan} = 0 \quad \text{on} \quad \partial\Omega_t \setminus \Gamma_t, \\ \left(\mu\left(\nabla\mathbf{u} + {}^t\nabla\mathbf{u}\right) - p\right)\mathbf{n} = -\left(\gamma\kappa + p^a\right)\mathbf{n} \quad \text{in} \quad \Gamma_t, \end{cases} \tag{2}$$

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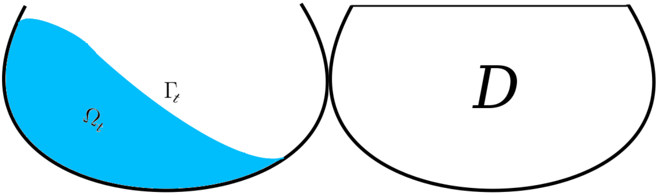

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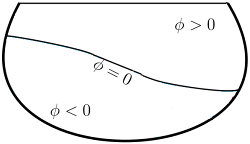
Kinematic boundary condition :

$$\left\{ (1,\mathbf{u}) \text{ is tangential to the free surface } (t,\Gamma_t),\ t \geq 0. \right. \tag{3}$$

Level set formulation to characterize the fluid domain

Ω_t is characterized by a continuous function ϕ such that



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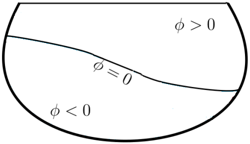

The kinematic boundary condition on the free surface is now replaced by

$$\begin{cases} \partial_t \phi + \tilde{\mathbf{u}} \cdot \nabla \phi = 0 & \forall (x,t) \in D \times [0,T], \\ \phi(0,x) = \phi_0(x) & \text{on} \quad D, \end{cases} \tag{4}$$

where $\tilde{\mathbf{u}}$ is a smooth field defined on D such that

$$\tilde{\mathbf{u}} = \mathbf{u} \quad \text{on } \Gamma_t,$$

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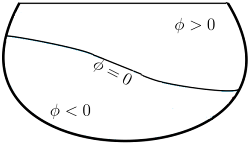

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Numerical resolution : Lagrange finite elements method

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At each iteration :

1. We move the fluid domain following the fluid velocity using the
   1. velocity extension (construction of velocity field of the level set advection equation)
   2. distancing (construction of the convenient initial condition for the level set advection equation)
   3. resolution of the advection equation (to find the level set corresponding to the advected domain).
2. We solve the time-discretized fluid equation on the advected domain using a convenient mesh adaptation.

- The level-set advection equation on the time sub-interval $[t^n, t^{n+1}]$ writes

$$\begin{cases} \partial_t \phi + \tilde{\mathbf{u}} \cdot \nabla \phi = 0 & \forall (x,t) \in D \times [t^n, t^{n+1}], \\ \phi(t^n, x) = \phi_{t^n}(x) & \text{on} \quad D, \end{cases} \tag{6}$$

where ϕ_{t^n} is a level-set function for the fluid domain Ω_{t^n} at time t^n.

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  in the vicinity of the free surface.
- Distancing is the construction of a level-set function $\phi_{t^n}(x)$ of Ω_{t^n} such that (7) is satisfied [Dapogny, Frey 2012] .

Method of characteristics

We use the method of characteristics

1. for the resolution of the level set advection equation. The solution is constant along the characteristics curves.
2. for the time discretization of the Navier–Stokes equation i.e. we use the fact that

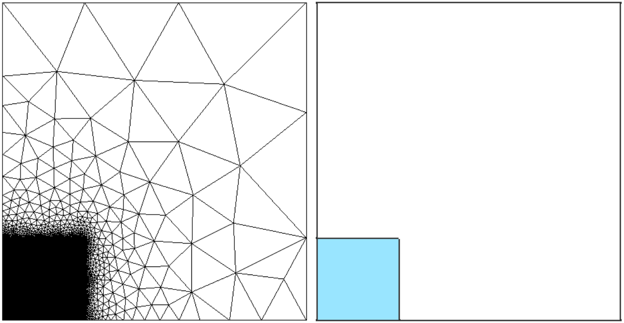
$$\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = \frac{d\mathbf{u}(\mathbf{X}(\mathbf{x},t;s),s)}{ds}\,|_{s=t},$$

where $\mathbf{X}(\mathbf{x},t;s)$ is the characteristic curve (parametrized by s) which cross $\mathbf{x}$ at time t.

$$\begin{cases} \rho\dfrac{\mathbf{u}^n(\mathbf{x}) - \mathbf{u}^{n-1}\circ\mathbf{X}^{n-1}(\mathbf{x})}{\Delta t} - \mu\Delta\mathbf{u}^n(\mathbf{x}) + \nabla p^n(\mathbf{x}) & = \rho\mathbf{f}^n \quad \text{in } \Omega_{t^n} \\ \operatorname{div}\mathbf{u}^n(\mathbf{x}) & = 0 \quad \text{in } \Omega_{t^n} \end{cases}$$

The characteristic equation is solved using a fourth order Runge-Kutta scheme.

Mesh adaptation



- The fluid free surface is explicitly discretized.
- The mesh is adapted such that the approximation, geometric and interpolation errors are controlled.
- Mesh elements outside the fluid (far from the free surface) are large since no value is of interest in this region.

Global scheme

Notations :

- $\mathcal{D}^n$ represents a convenient meshing of the computational domain D at time t^n.
- $\mathcal{T}^n$ is a subset of $\mathcal{D}^n$ and represents a convenient meshing of the fluid domain Ω_{t^n} at time t^n

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Scheme

1. *At time* $t^0 = 0$: $\mathcal{T}^0 \subset \mathcal{D}^0$ *characterizing* Ω_{t^0}, $\mathbf{u}^0$.
2. *For* $n = 0, ..., N-1$,

	mesh	*input*	*output*
Velocity extension	$(\mathcal{D}^n \setminus \mathcal{T}^n) \cup \Gamma^n$	$\mathbf{u}^n$	$\tilde{\mathbf{u}}^n$
Distancing	$\mathcal{D}^n$	$(\mathcal{D}^n, \mathcal{T}^n)$	ϕ^n
Advection	$\mathcal{D}^n$	$(\tilde{\mathbf{u}}^n, \phi^n)$	ϕ^{n+1}
Mesh adaptation	$\mathcal{D}^n$	$(\phi^n, \mathcal{D}^n)$	$\mathcal{D}^{n+1}$
Interpolation	$\mathcal{D}^{n+1}$	$\tilde{\mathbf{u}}^n$	$\tilde{\mathbf{u}}^n$
NS resolution	$\mathcal{T}^{n+1}$	$\tilde{\mathbf{u}}^n$	$(\mathbf{u}^{n+1}, p^{n+1})$

3. *We return* $\mathbf{u}^N, p^N, \phi^N, \mathcal{D}^N$.

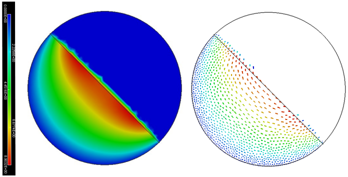
Related paper :

- Pascal Frey, Dena Kazerani, Thi Thanh Mai Ta. An adaptive numerical scheme for solving incompressible two-phase and free-surface flows. In revision for International Journal for Numerical Methods in Fluids.

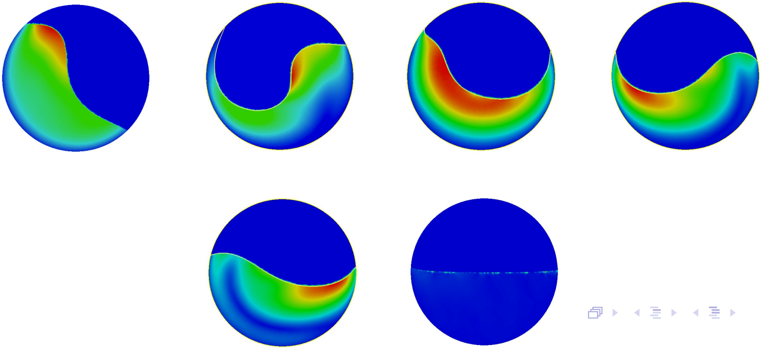
Plan

Viscous fluid in the unit circular domain

Parameters : $\rho = 1,\ \mu = 1,\ \gamma = 7.2,\ \alpha = 10,\ p^a = 0.$
Data : $\mathbf{f} = (0, -100)$ and the initial solution is given by :



Space and time step : $\Delta t \simeq 10^{-3}$, $h_{min} = h_{max} = 0.3$ and **Final time :** $T = 4.435$.



Dam break test cases

The dam break test cases considered here are the same as in

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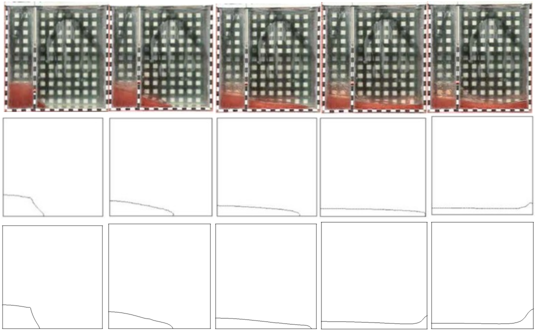
Differences	**Our framework**	**Reference's framework**
Mathematical problem	bi-fluid problem	free-surface fluid
Fluid equation	computational domain	fluid domain
Meshing	homogenous fixed meshing	mobile unstructured meshing
Advection equation	ETILT	method of characteristics
Time step	$\Delta t = 0.001$	5/10 times larger

- We use the same friction law as well as the same turbulent model (for the dam break with water problem) as in this reference.

Dam break with shampoo

Parameters : $\rho = 1024$, $\mu = 8$, $\gamma = 0.07$, $p^a = 0$ and $\alpha = 10^{-2}$ (computed following the law $\alpha = \frac{\rho|\mathbf{U}|}{C_f^2}$ with $C_f = 190$ and $|\mathbf{U}| = 0.4$).
Data : $\mathbf{f} = (0, -9.8)$.
Domain dimensions : Computational domain is $0.42m \times 0.44m$, fluid domain is $0.114m \times 0.114m$.
Time step : $\Delta t = 0.01$ or the first ten iterations and $\Delta t = 0.02$ for other iterations.
Final time : $T = 0.5$.
Space step : $h_{min} = 0.0009$, $h_{max} = 1$ and $hgrad = 2.5$.



$t = 0.1$ $t = 0.2$ $t = 0.3$ $t = 0.4$ $t = 0.5$

Dam break with water

Parameters : $\rho = 1000$, $\mu = 0.001$, $\alpha = 10^{-6}$, $p^a = 0$ and $\gamma = 0.07$.
Data : $\mathbf{f} = (0, -0.98)$.
Specificity : Reynolds number is high $\rightsquigarrow$ turbulent effects $\rightsquigarrow$ energy dissipation $\rightsquigarrow$ we correct the model by modifying the viscosity as

$$\mu_{mod} = \min\left(\mu + l_{mix}^2 \rho \sqrt{\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u})/2}; \mu_{max}\right),$$

where $l_{mix} = C_t \; h_{UGN}$ such that C_t is a modelling parameter, h_{UGN} is a characteristic element size and μ_{max} is a cut-off value. We set here

$$C_t = 3.57, \quad h_{UGN} = hmin = 9 \times 10^{-4}, \quad \mu_{max} = 1.5.$$

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In practice : we take $\mu = 0.001$ for the first iteration of the algorithm. Then, we take the cut-off value μ_{max} for viscosity i.e. 1.5.

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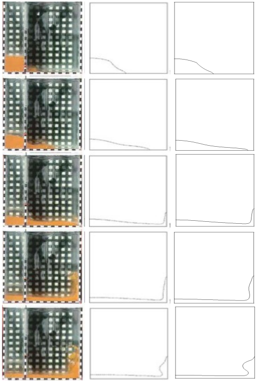
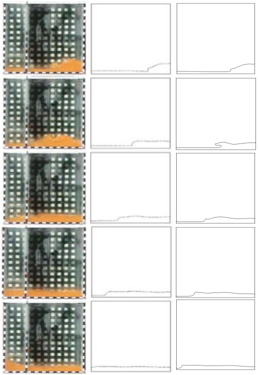
Time step : $\Delta t = 0.001$ for the first iteration of the algorithm and $\Delta t = 0.005$ for others.
Final time : T= 1.5.
Space step : $hmax = 1$ and $hgrad = 2.5$.

Dam break with water

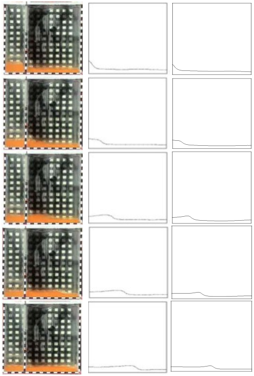
$t = 0.1 \rightsquigarrow 0.5$



$t = 0.6 \rightsquigarrow 1.0$



$t = 1.1 \rightsquigarrow 1.5$



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Conclusion and perspectives

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Perspectives :

- Modelling the atmospheric pressure (du to the external fluid) for the simulation of the free-surface Navier–Stokes
- Some error estimates for the presented numerical scheme (Numerical test cases seem to be very sensitive to the variation of the time step)
- Extension to complex fluids (viscoelastic, viscoplastic, ...)
- Extension to sloshing problems
- Comparison between the numerical solution of the free-surface Navier–Stokes problem and shallow water models

Thank you for your attention !