Etude d'un système couplé des équations de St-Venant avec un modèle de couche limite pour la modélisation de l'érosion

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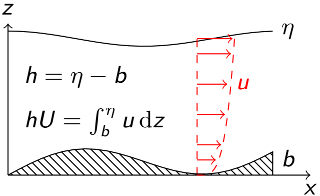
5ème école EGRIN

Cargèse, 02 Juin 2017

Plan de l'exposé

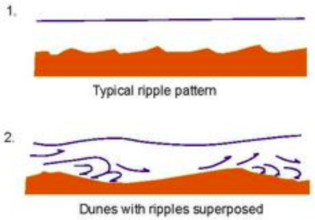
Couplage St Venant - Exner

$$\partial_t h + \partial_x(hU) = 0,$$
$$\partial_t(hU) + \partial_x(hU^2 + gh^2/2) = -gh\partial_x b - \tau_b,$$
$$\partial_t b + \partial_x q_s(\tau_b - \tau_c) = 0, \quad q_s = c_{MPM}(\tau_b - \tau_c)_+^{3/2}.$$



$$\tau_b := \nu \partial_z u|_{z=b}$$

- Poiseuille : $\tau_b = \frac{3\nu}{h} U$
- Darcy-Weisbach : $\tau_b = \frac{f}{8} U^2$
- Manning : $\tau_b = \frac{gn^2}{h^{1/3}} U^2$



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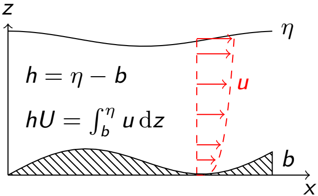

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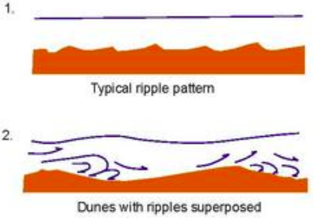


$$\left.\begin{aligned} b &= b_0 e^{\sigma t + ikx}, \\ q_s &= q_{s0} e^{\sigma t + ik(x+\ell)} \end{aligned}\right\} \Rightarrow \mathrm{Re}(\sigma) = \frac{q_{s0} k}{b_0} \sin(k\ell)$$

St Venant donne $\ell = 0$

Kennedy (1963), Fowler (2001)

Charru (2012) : Instabilités hydrodynamiques

Plan de l'exposé

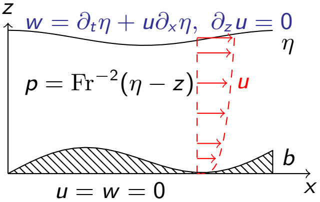
Couplage St Venant - VonKarman (1)

M. Legrand (2016) : Etude mathématique de modèles de couches visqueuses pour des écoulements naturels

$$\boxed{\varepsilon = H_0/L_0,\ \mathrm{Re} = H_0 U_0/\nu,\ \mathrm{Fr}^2 = U_0^2/gH_0}$$

$$\partial_x u + \partial_z w = 0,$$

$$\partial_t u + \partial_x u^2 + \partial_z uw = -\partial_x p + (\varepsilon \mathrm{Re})^{-1}\partial_z^2 u,$$



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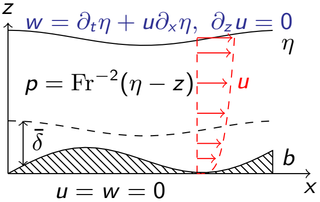

Transformation de Prandtl : $\bar{\delta} := (\varepsilon \mathrm{Re})^{-1/2}$

$$t = \bar{t}, \quad x = \bar{x}, \quad z = b + \bar{\delta}\bar{z},$$

$$u = \bar{u}, \quad p = \bar{p}, \quad w = b' u + \bar{\delta}\bar{w},$$

$$\partial_{\bar{t}}\bar{u} + \bar{u}\partial_{\bar{x}}\bar{u} + \bar{w}\partial_{\bar{z}}\bar{u} = -\partial_{\bar{x}}\bar{p} - \mathrm{Fr}^{-2}b' + \partial_{\bar{z}}^2\bar{u}$$

$$\partial_{\bar{t}}\bar{u}_e + \bar{u}_e\partial_{\bar{x}}\bar{u}_e = -\partial_{\bar{x}}\bar{p} - \mathrm{Fr}^{-2}b' + \partial_{\bar{z}}^2\bar{u}|_{\bar{z}=\bar{\eta}}$$

$$\boxed{\partial_{\bar{t}}(\delta_1\bar{u}_e) + \delta_1\bar{u}_e\partial_{\bar{x}}\bar{u}_e + \partial_{\bar{x}}(\delta_2\bar{u}_e^2) = \partial_{\bar{z}}\bar{u}|_{\bar{z}=0} + \bar{\eta}\partial_{\bar{z}}^2\bar{u}|_{\bar{z}=\bar{\eta}}}$$

$$u_e(t,x) = \bar{u}_e(\bar{t},\bar{x}) := \bar{u}(\bar{t},\bar{x},\bar{\eta})$$

$$\delta_1 := \int_0^{\bar{\eta}} \left(1 - \frac{\bar{u}}{\bar{u}_e}\right) \mathrm{d}\bar{z}$$

$$\delta_2 := \int_0^{\bar{\eta}} \frac{\bar{u}}{\bar{u}_e}\left(1 - \frac{\bar{u}}{\bar{u}_e}\right) \mathrm{d}\bar{z},$$

Couplage St Venant - VonKarman (2)

$$\partial_t h + \partial_x \int_b^\eta u \,\mathrm{d}z = 0,$$

$$\partial_t \int_b^\eta u \,\mathrm{d}z + \partial_x \int_b^\eta u^2 \,\mathrm{d}z = -h\partial_x p - \bar{\delta}^2 \partial_z u|_{z=b},$$

$$\partial_t(\delta_1 u_e) + \partial_x(\delta_1 u_e^2 + \delta_2 u_e^2) = u_e \partial_x(\delta_1 u_e) + \bar{\delta}\partial_z u|_{z=b} + \bar{\delta} h \partial_z^2 u|_{z=\eta}.$$

$$\boxed{\int_b^\eta u \,\mathrm{d}z = hu_e - \bar{\delta}\delta_1 u_e, \quad \int_b^\eta u^2 \,\mathrm{d}z = hu_e^2 - \bar{\delta}(\delta_1 + \delta_2)u_e^2}$$

Fermeture $\mathcal{H} := \dfrac{\delta_1}{\delta_2}, \quad \bar{\delta}\partial_z u|_{z=b} = \partial_{\bar{z}} \bar{u}|_{\bar{z}=0} := \dfrac{f_2 \bar{u}_e}{\delta_2} = \dfrac{f_2 \mathcal{H} \bar{u}_e}{\delta_1}$

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Système final

$$\begin{cases} \partial_t h + \partial_x(h u_e - \bar{\delta}\delta_1 u_e) = 0, \\ \partial_t(h u_e) + \partial_x \left(h u_e^2 + \dfrac{g h^2}{2} \right) = -g h \partial_x b + u_e \partial_x(\bar{\delta}\delta_1 u_e) + \bar{\delta}^2 h \partial_z^2 u|_{z=\eta}, \\ \partial_t(\delta_1 u_e) + \partial_x \left((1 + \dfrac{1}{\mathcal{H}}) \delta_1 u_e^2 \right) = u_e \partial_x(\delta_1 u_e) + \dfrac{f_2 \mathcal{H}}{\delta_1 u_e} u_e^2 + \bar{\delta} h \partial_z^2 u|_{z=\eta}. \end{cases}$$

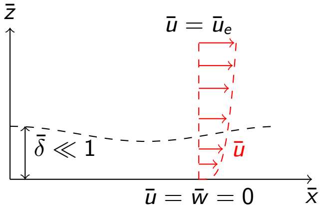
Solutions de Blasius et Falkner-Skan

Hypothèses : $\bar{\delta} \ll 1,\ \partial_{\bar{t}} = 0,\ \partial^2_{\bar{z}} \bar{u}|_{\bar{z}=\bar{\eta}} = 0$

$$\partial_{\bar{x}} \bar{u} + \partial_{\bar{z}} \bar{w} = 0,$$

$$\bar{u}\partial_{\bar{x}} \bar{u} + \bar{w}\partial_{\bar{z}} \bar{u} = \bar{u}_e \partial_{\bar{x}} \bar{u}_e + \partial^2_{\bar{z}} \bar{u}$$

$$\bar{u} = \bar{u}_e \quad \text{quand} \quad \bar{z} \to \infty$$



Schlichting (1968) : Boundary-layer theory

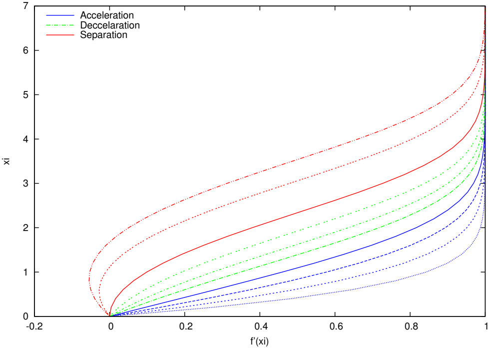
Equation de FS : $\bar{u}(\bar{x}, \bar{z}) := \bar{u}_e(\bar{x}) f'(\xi)$

$$\boxed{f''' + ff'' + \beta(1 - f') = 0}$$

$$f(0) = f'(0) = 0,\ f'(\infty) = 1,$$

$$\text{et } -0.1988 \leq \beta \leq 2,$$

$$\delta_1^2 \partial_x u_e = \beta \left(\int_0^\infty (1 - f')\, \mathrm{d}\xi \right)^2$$



Fermeture basée sur Blasius et FS

$$\Lambda_1 := \delta_1^2 \partial_x u_e = \beta \left(\int_0^\infty (1 - f') \, \mathrm{d}\xi \right)^2$$

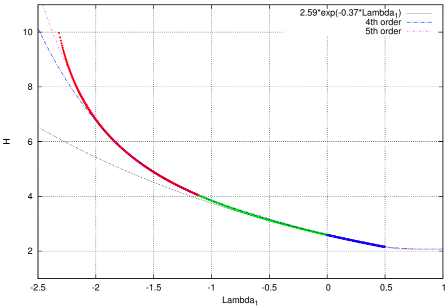
$$\mathcal{H} = \int_0^\infty (1 - f') \, \mathrm{d}\xi \Big/ \int_0^\infty f'(1 - f') \, \mathrm{d}\xi$$

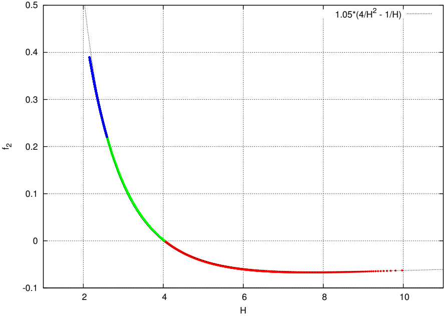
$$f_2 = f''(0) \int_0^\infty f'(1 - f') \, \mathrm{d}\xi$$

Blasius : $\Lambda_1 = 0, \mathcal{H} = 2.59, f_2 = 0.22$

$$\mathcal{H} = \begin{cases} 2.59 e^{-0.37\Lambda_1} & \text{si } \Lambda_1 < 0.6, \\ 2.074 & \text{si } \Lambda_1 \geq 0.6, \end{cases}$$

$$f_2 = 1.05 \left(\frac{4}{\mathcal{H}^2} - \frac{1}{\mathcal{H}} \right).$$





Plan de l'exposé

Hyperbolicité & schéma numérique

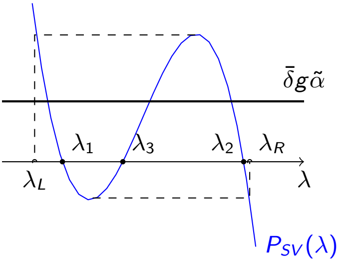
Cordier et al (2011), P. Ung (2016)

Polynôme caractéristique

$$P(\lambda) = (\tilde{\beta} - u_e - \lambda)\left((u_e - \lambda)^2 - gh\right) - \bar{\delta} g \tilde{\alpha}$$

$$\tilde{\alpha} = (1 + \frac{1}{\mathcal{H}})\delta_1 u_e - \frac{\delta_1 u_e^2}{\mathcal{H}^2}\frac{\partial \mathcal{H}}{\partial u_e}$$

$$\tilde{\beta} = (1 + \frac{1}{\mathcal{H}})u_e - \frac{\delta_1 u_e^2}{\mathcal{H}^2}\frac{\partial \mathcal{H}}{\partial(\delta_1 u_e)}$$



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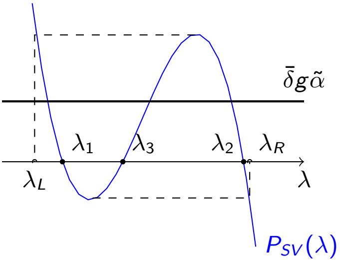
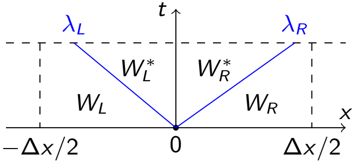


Schéma de type Godunov

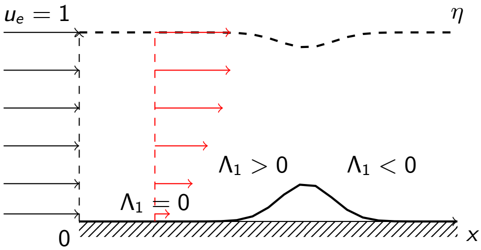
$$W_j^{n+1} = W_j^n - \frac{\Delta t}{\Delta x}\left(F^L_{j+1/2} - F^R_{j-1/2}\right) + S(W_j^{n,n+1}),$$

$$F^L(W_L, W_R) = F(W_L) + \lambda_L(W_L^* - W_L),$$

$$F^R(W_L, W_R) = F(W_R) - \lambda_R(W_R - W_R^*).$$



Ecoulement fluvial sur une bosse gaussienne : SVVK vs SVML



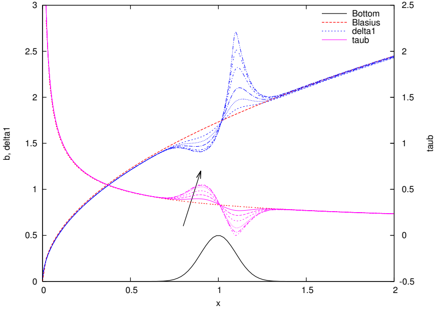
Blasius : $\Lambda_1 = 0$

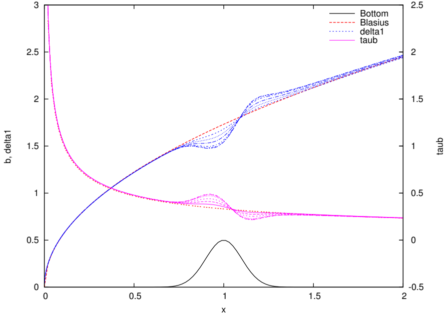
$$\mathcal{H} = 2.59, f_2 = 0.22$$

$$\delta_1 = 1.73\sqrt{x}$$

$$\tau_b = 0.322/\sqrt{x}$$

Audusse et al (2011)





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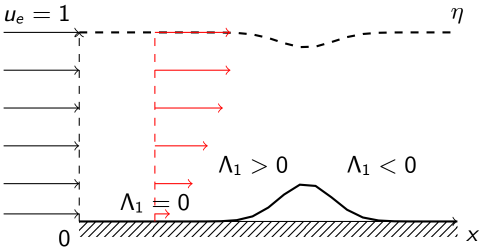

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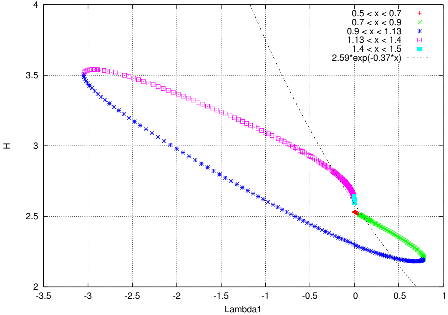
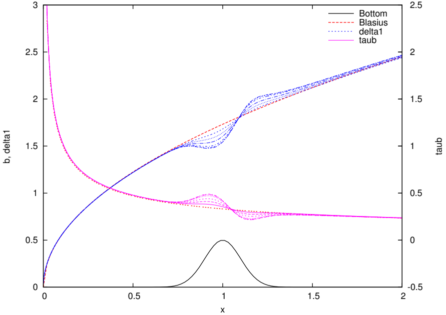




Conclusion

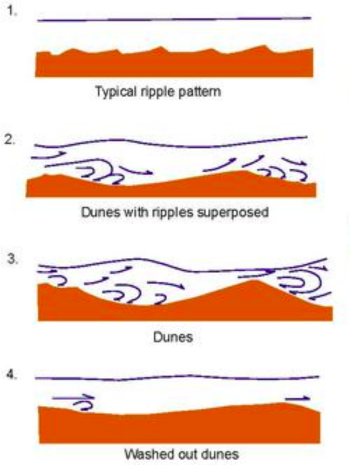
Conclusion : le couplage St Venant - VonKarman permet de

- retrouver le déphasage entre frottement et le fond,
- capturer une éventuelle recirculation de l'écoulement,
- favoriser le développement de dunes.

Difficultés et perspective :

- le modèle nécessite des bonnes fermetures pour $\mathcal{H}$, f_2,
- il est important de modéliser la transition de FS à Poiseuille, rajouter la turbulence (longueur de mélange ?), et coupler avec Exner

$$\partial_t b + \partial_x q_s(\tau_b - \tau_c - \gamma\partial_x b), \quad \gamma > 0.$$



https://en.wikipedia.org/wiki/Bedform