Hyperbolicity of the Layerwise Discretized Shallow Water equations

The bilayer case

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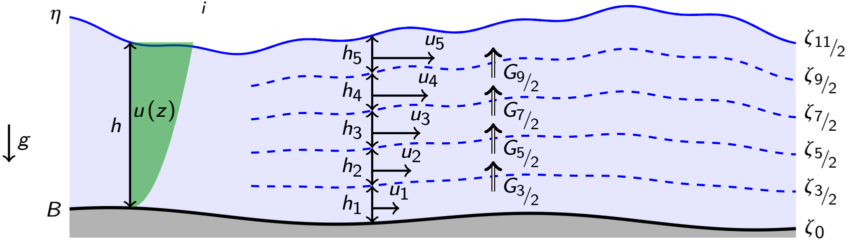
Layerwise Discretized Shallow Water model:

[Audusse, Bristeau, Perthame and Sainte-Marie'11]

for $i \in [\![1, L]\!]$

$$(SW_L) \quad \begin{cases} \partial_t h_i + \partial_x (h_i u_i) + \partial_y (h_i v_i) = [G]_{i-1/2}^{i+1/2} \\ \partial_t (h_i u_i) + \partial_x \left(h_i u_i^2 + \frac{g}{2} h_i h\right) + \partial_y (h_i u_i v_i) = -g h_i \partial_x B + [uG]_{i-1/2}^{i+1/2} \\ \partial_t (h_i v_i) + \partial_x (h_i v_i u_i) + \partial_y \left(h_i v_i^2 + \frac{g}{2} h_i h\right) = -g h_i \partial_y B + [vG]_{i-1/2}^{i+1/2} \end{cases}$$

with $h_i = \alpha_i h$ and $\sum_i \alpha_i = 1$.



[Audusse, Bristeau, Pelanti and Sainte-Marie'11] with variable density.

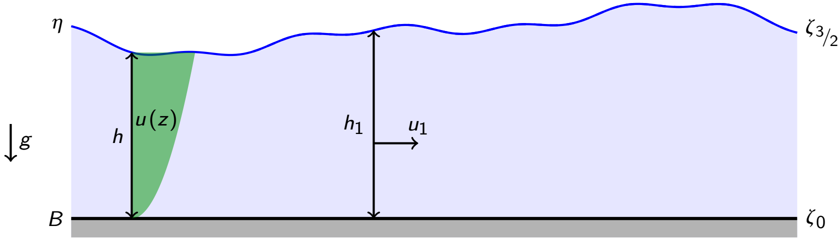
[Bristeau, Guichard, Di Martino and Sainte-Marie'16] with viscous terms.

[Fernandez-Nieto, Parisot, Penel and Sainte-Marie] with non-hydrostatic terms.

Layerwise Discretized Shallow Water model:

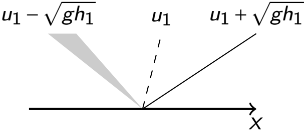
[Audusse, Bristeau, Perthame and Sainte-Marie'11]

$$(SW_1) \quad \begin{cases} \partial_t h_1 + \partial_x (h_1 u_1) = 0 \\ \partial_t (h_1 u_1) + \partial_x \left(h_1 u_1^2 + \frac{g}{2} h_1^2\right) = 0 \\ \partial_t (h_1 v_1) + \partial_x (h_1 v_1 u_1) = 0 \end{cases}$$



- **Strictly hyperbolic** equations
- Admissible shock define to ensure the **mechanical energy dissipation** : $E = \frac{h_1}{2}\left(u_1^2 + v_1^2\right) + \frac{g}{2} h_1^2$

$$\partial_t E + \partial_x \left(\left(\frac{u_1^2 + v_1^2}{2} + g h_1 \right) h_1 u_1 \right) \leq 0$$



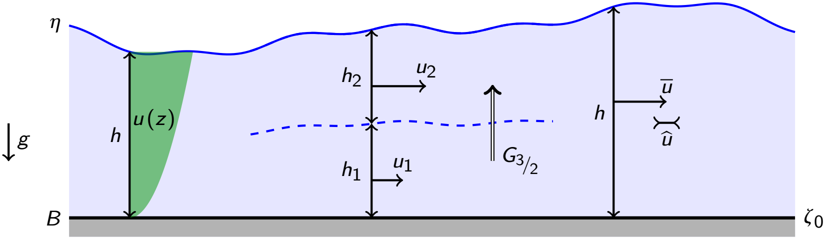
Dambreak SW_1

Layerwise Discretized Shallow Water model:

[Audusse, Bristeau, Perthame and Sainte-Marie'11]

$$(SW_2)\quad h_1 = h_2,\quad u_{3/2} = \frac{u_1+u_2}{2},\quad v_{3/2} = \frac{v_1+v_2}{2}$$

$$\begin{cases} \partial_t h_1 + \partial_x (h_1 u_1) = -G_{3/2} \\ \partial_t h_2 + \partial_x (h_2 u_2) = G_{3/2} \\ \partial_t (h_1 u_1) + \partial_x \left(h_1 u_1^2 + \frac{g}{2} h_1 h\right) = -u_{3/2} G_{3/2} \\ \partial_t (h_2 u_2) + \partial_x \left(h_2 u_2^2 + \frac{g}{2} h_2 h\right) = u_{3/2} G_{3/2} \\ \partial_t (h_1 v_1) + \partial_x (h_1 v_1 u_1) = -v_{3/2} G_{3/2} \\ \partial_t (h_2 v_2) + \partial_x (h_2 v_2 u_2) = v_{3/2} G_{3/2} \end{cases} \Leftrightarrow \begin{cases} \partial_t h + \partial_x (h\overline{u}) = 0 \\ \partial_t (h\overline{u}) + \partial_x \left(h\left(\overline{u}^2 + \widehat{u}^2\right) + \frac{g}{2} h^2\right) = 0 \\ \partial_t \widehat{u} + \partial_x (\widehat{u}\,\overline{u}) = 0 \\ \partial_t (h\overline{v}) + \partial_x (h(\overline{u}\,\overline{v} + \widehat{u}\,\widehat{v})) = 0 \\ \partial_t \widehat{v} + \widehat{u}\partial_x \overline{v} + \overline{u}\partial_x \widehat{v} = 0 \\ \overline{u} = \frac{u_1+u_2}{2} \quad \widehat{u} = \frac{u_2-u_1}{2} \\ \overline{v} = \frac{v_1+v_2}{2} \quad \widehat{v} = \frac{v_2-v_1}{2} \end{cases}$$



[Teshukov'07, Richard and Gavrilyuk'12] shear model.

[Castro and Lannes'14] with non-hydrostatic terms.

Bi-layer model:

$$\begin{cases} \partial_t h & + \partial_x (h\overline{u}) & = 0, \\ \partial_t (h\overline{u}) & + \partial_x \left(h\overline{u}^2 + h\left(\widehat{u}^2 + \frac{g}{2}h\right)\right) & = 0, \\ \partial_t \widehat{u} & + \partial_x (\widehat{u}\,\overline{u}) & = 0, \\ \partial_t (h\overline{v}) & + \partial_x (h(\overline{u}\,\overline{v} + \widehat{u}\,\widehat{v})) & = 0, \\ \partial_t \widehat{v} & + \widehat{u}\partial_x \overline{v} + \overline{u}\partial_x \widehat{v} & = 0 \end{cases}$$

- Admissible shock define to ensure the **mechanical energy dissipation** :
  $E = \frac{h}{2}\left(\overline{u}^2 + \widehat{u}^2 + \overline{v}^2 + \widehat{v}^2\right) + \frac{g}{2}h^2$

$$\partial_t E + \partial_x \left(\left(\frac{\overline{u}^2 + 3\widehat{u}^2 + \overline{v}^2 + \widehat{v}^2}{2} + gh\right) h\overline{u} + h\overline{v}\widehat{v}\widehat{u}\right) \leq 0$$

Full-Euler model:

$$\begin{cases} \partial_t \rho & + \partial_x (\rho u) & = 0, \\ \partial_t (\rho u) & + \partial_x \left(\rho u^2 + p\right) & = 0, \\ \partial_t (\rho e) & + \partial_x ((\rho e + p) u) & = 0, \\ \partial_t (\rho v) & + \partial_x (\rho u v) & = 0, \end{cases}$$

- Admissible shock define to ensure the **entropy dissipation** :

$$\partial_t \eta + \partial_x (\eta u) \leq 0$$

- 1D analogous to the full-Euler equations **except at the shock**.

Bi-layer model:

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- Admissible shock define to ensure the **mechanical energy dissipation** : $E = \frac{h}{2}\left(\overline{u}^2 + \widehat{u}^2 + \overline{v}^2 + \widehat{v}^2\right) + \frac{g}{2}h^2$

$$\partial_t E + \partial_x \left(\left(\frac{\overline{u}^2 + 3\widehat{u}^2 + \overline{v}^2 + \widehat{v}^2}{2} + gh\right) h\overline{u} + h\overline{v}\widehat{v}\widehat{u}\right) \leq 0$$

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- Admissible shock define to ensure the **entropy dissipation** :

$$\partial_t \eta + \partial_x (\eta u) \leq 0$$

- 1D analogous to the full-Euler equations **except at the shock**.
- 2D NOT analogous : **non-conservative products, coalescence, resonance.**

1D Bi-layer model:

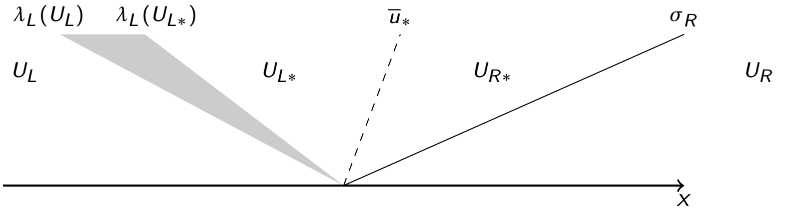
$$\begin{cases} \partial_t h + \partial_x (h\overline{u}) = 0, \\ \partial_t (h\overline{u}) + \partial_x \left(h\overline{u}^2 + h\left(\widehat{u}^2 + \dfrac{g}{2} h\right)\right) = 0, \\ \partial_t \widehat{u} + \partial_x (\widehat{u}\,\overline{u}) = 0, \end{cases} \qquad \text{We set } U = \begin{pmatrix} h \\ h\overline{u} \\ \widehat{u} \end{pmatrix}$$

Proposition : hyperbolicity of $(1D - SW_2)$

For physical solution, i.e. $h > 0$, the 1D bilayer model (SW_2) is **strictly hyperbolic**.
More precisely, the eigenvalues are given by

$$\lambda_L = \overline{u} - \sqrt{gh + 3\widehat{u}^2} \quad < \quad \lambda_* = \overline{u} \quad < \quad \lambda_R = \overline{u} + \sqrt{gh + 3\widehat{u}^2}.$$

In addition, the λ_L-wave and the λ_R-wave are **genuinely nonlinear**,
whereas the λ_*-wave is **linearly degenerate**.



Proposition : admissible shock of $(1D-SW_2)$

We denote by σ_k the speed of the λ_k-shock.
Assuming that the water depth h is positive, the following properties are equivalent:

i) The mechanical energy $E = \frac{g}{2}h^2 + \frac{h}{2}\left(\overline{u}^2 + \widehat{u}^2\right)$ is **decreasing** through a shock, i.e.

$$-\sigma_k [E] + \left[\left(\frac{\overline{u}^2 + 3\widehat{u}^2}{2} + gh\right)h\overline{u}\right] < 0.$$

ii) the shock is **compressive**, i.e. we have

$$-\sigma_k \left[h^2\right] + \left[h^2\overline{u}\right] > 0 \qquad \text{or} \qquad -\sigma_k \left[h\overline{u}^2\right] + \left[h\overline{u}^3\right] < 0 \qquad \text{or} \qquad -\sigma_k \left[h\widehat{u}^2\right] + \left[h\widehat{u}^2\overline{u}\right] > 0.$$

iii) the **Lax entropy condition** is satisfied

$$\lambda_L(U_{L*}) < \sigma_L < \lambda_L(U_L) \qquad \text{and} \qquad \lambda_R(U_R) < \sigma_R < \lambda_R(U_{R*}).$$

Remark : It is NOT a corollary of the classical theorem [Godlewsli, Raviart'96] since the mechanical energy E (acting as the mathematical entropy) is **not a convexe function** of the conserved variable U.

Theorem : Riemann problem of $(1D - SW_2)$

Consider the initial condition $U(0,x) = \begin{cases} U_L = (h_L, \overline{u}_L, \widehat{u}_L)^t \in \mathbb{R}_+^* \times \mathbb{R}^2 & \text{if } x < 0, \\ U_R = (h_R, \overline{u}_R, \widehat{u}_R)^t \in \mathbb{R}_+^* \times \mathbb{R}^2 & \text{if } x \geq 0. \end{cases}$

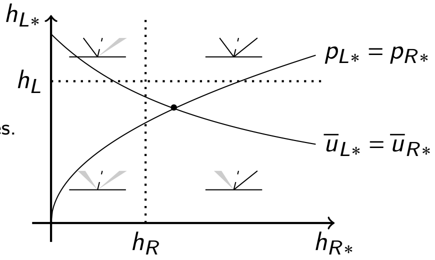
If the following condition is fulfilled : $\overline{u}_R - \overline{u}_L < \mu^r(U_L) + \mu^r(U_R)$ with

$$\mu^r(U) = \sqrt{gh + 3\widehat{u}^2} + \frac{gh}{\sqrt{3}\widehat{u}} \log\left(\sqrt{1 + \frac{3\widehat{u}^2}{gh}} + \widehat{u}\sqrt{\frac{3}{gh}}\right)$$

then there exists a **unique selfsimilar** solution $U \in (L^\infty(\mathbb{R}_+^* \times \mathbb{R}))^3$ to the 1D Riemann problem (SW_2) satisfying the **mechanical energy dissipation**.
In addition the water depth h is **strictly positive** for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

Sketch of proof :

1. Determine the rarefaction and shock curves.
2. Conclude by monotony.



2D Bi-layer model:

$$\begin{cases} \partial_t h + \partial_x (h\overline{u}) = 0, \\ \partial_t (h\overline{u}) + \partial_x \left(h\left(\overline{u}^2 + \widehat{u}^2\right) + \frac{g}{2} h^2 \right) = 0, \\ \partial_t \widehat{u} + \partial_x (\widehat{u}\,\overline{u}) = 0, \\ \partial_t (h\overline{v}) + \partial_x (h(\overline{u}\,\overline{v} + \widehat{u}\,\widehat{v})) = 0, \\ \partial_t \widehat{v} + \widehat{u}\partial_x \overline{v} + \overline{u}\partial_x \widehat{v} = 0, \end{cases}$$

We set $V = \begin{pmatrix} h \\ h\overline{u} \\ \widehat{u} \\ h\overline{v} \\ \widehat{v} \end{pmatrix}$

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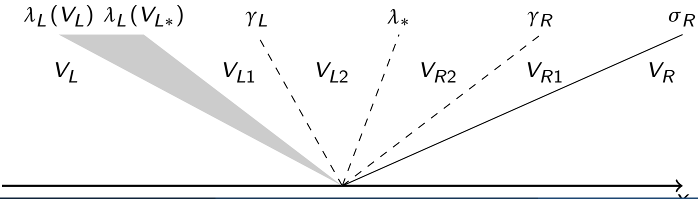

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We set $V = \begin{pmatrix} h \\ h\overline{u} \\ \widehat{u} \\ h\overline{v} \\ \widehat{v}\end{pmatrix}$, where the first three components $\left.\begin{matrix} h \\ h\overline{u} \\ \widehat{u}\end{matrix}\right\} U$ $\quad (1D-SW_2)$

Proposition : hyperbolicity of $(2D-SW_2)$

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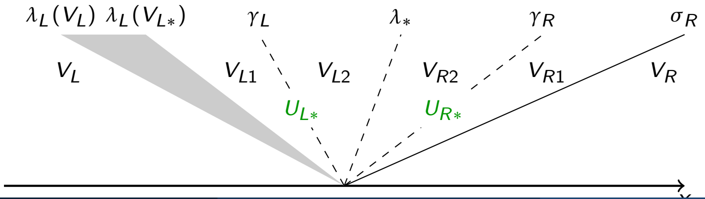

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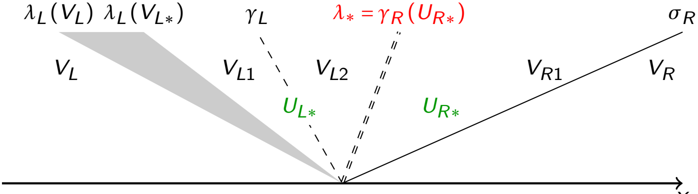

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$$\text{We set } V = \begin{pmatrix} h \\ h\overline{u} \\ \widehat{u} \\ h\overline{v} \\ \widehat{v} \end{pmatrix} \quad \left.\begin{matrix} h \\ h\overline{u} \\ \widehat{u} \end{matrix}\right\} U \qquad (1D - SW_2)$$

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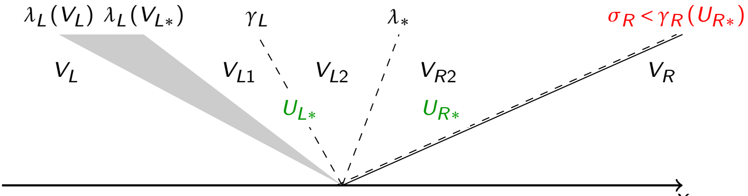
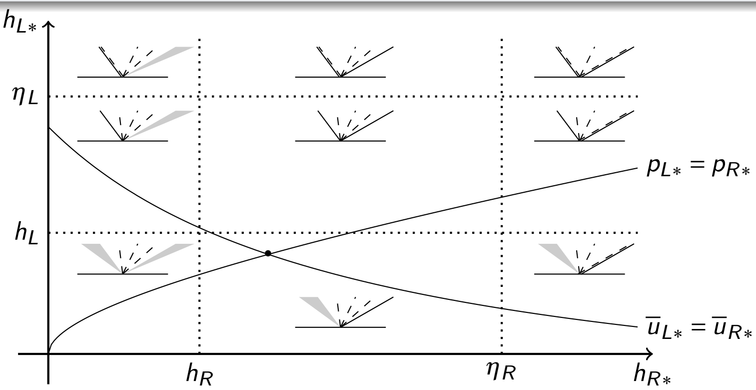


Proposition : coalescence

Let us define the following polynomial function for $X > h$

$$P(U\,;X) := \frac{1}{2}\left(1+\frac{X}{h}\right)gh + \left(1+\frac{X}{h}+\left(\frac{X}{h}\right)^2-\left(\frac{X}{h}\right)^3\right)\widehat{u}^2.$$

If $\widehat{u}\neq 0$, let $\eta(U)$ be the **unique real root** larger than h.
The coalescence occurs if and only if $h_{k*} \geq \eta_k := \eta(U_k)$.



In practice, it is enough to test $P(U_k; h_{k*}) \begin{cases} > 0 & \text{no-coalescence} \\ < 0 & \text{coalescence.} \end{cases}$

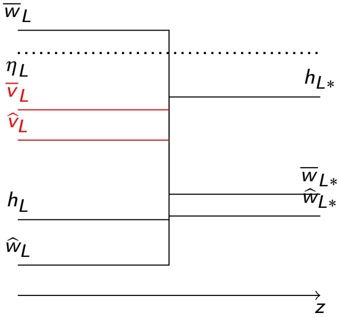
[Lagoutière, Seguin, Takahashi'08] **Path-independent** non-conservative product definition.

Definition : non-conservative products

Through a λ_k-shock, the following **jump conditions** hold:

$$\begin{cases} \partial_t h + \partial_x (h\overline{u}) = 0, \\ \partial_t (h\overline{u}) + \partial_x \left(h\left(\overline{u}^2 + \widehat{u}^2\right) + \frac{g}{2} h^2 \right) = 0, \\ \partial_t \widehat{u} + \partial_x (\widehat{u}\,\overline{u}) = 0, \\ \partial_t (h\overline{v}) + \partial_x (h(\overline{u}\,\overline{v} + \widehat{u}\,\widehat{v})) = 0, \\ \partial_t \widehat{v} + \widehat{u}\partial_x \overline{v} + \overline{u}\partial_x \widehat{v} = 0, \end{cases} \Leftrightarrow \begin{cases} \sigma_k [h] = [h\overline{u}] \\ \sigma_k [h\overline{u}] = \left[h\left(\overline{u}^2 + \widehat{u}^2\right) + \frac{g}{2} h^2 \right] \\ \sigma_k [\widehat{u}] = [\widehat{u}\overline{u}] \\ \sigma_k [h\overline{v}] = [h(\overline{u}\,\overline{v} + \widehat{u}\widehat{v})] \\ \left[h\sqrt{(\overline{u} - \sigma_k)^2 - \widehat{u}^2}\ \widehat{v} \right] = 0 \qquad \text{if } \textbf{no-coalescence}. \end{cases}$$

Arguments : $z = x - \sigma_k t$, $\overline{w} = \overline{u} - \sigma_k$ and $\widehat{w} = \widehat{u}$



[Lagoutière, Seguin, Takahashi'08] **Path-independent** non-conservative product definition.

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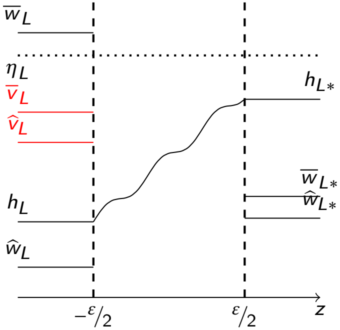

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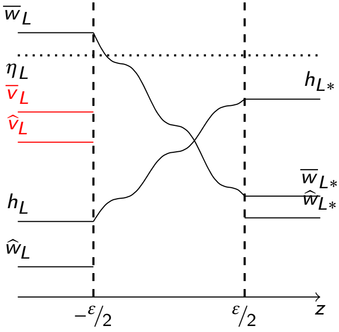

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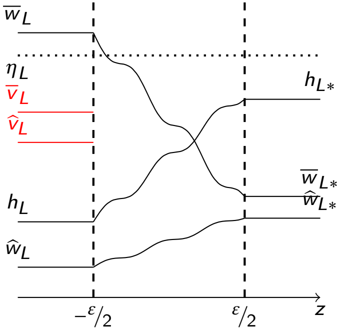

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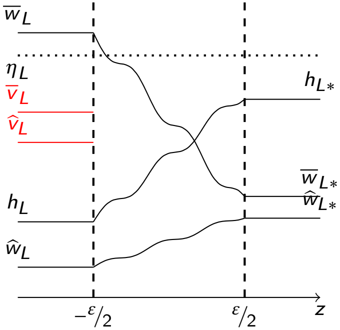

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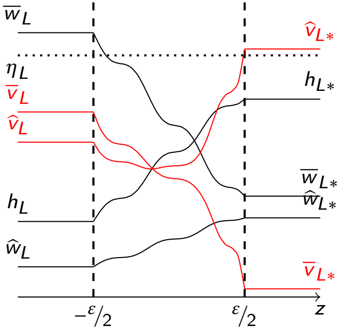

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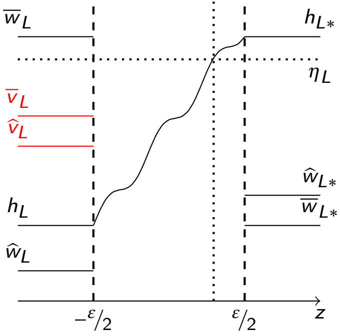

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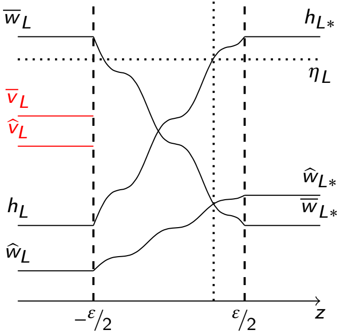

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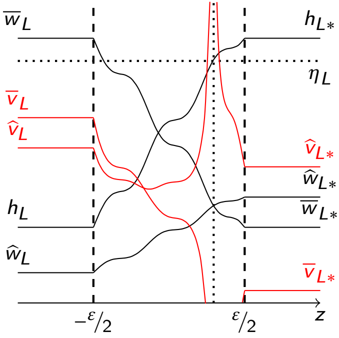


Proposition : admissible shock of $(2D-SW_2)$

We denote by σ_k the speed of the λ_k-shock.
Assuming that the water depth h is positive, the following properties are equivalent:

i) The mechanical energy $E=\frac{g}{2}h^2+\frac{h}{2}\left(\overline{u}^2+\widehat{u}^2+\overline{v}^2+\widehat{v}^2\right)$ is **decreasing** through a shock, i.e.

$$-\sigma_k\,[E]+\left[\left(\frac{\overline{u}^2+3\widehat{u}^2+\overline{v}^2+\widehat{v}^2}{2}+gh\right)h\overline{u}+h\overline{v}\widehat{v}\widehat{u}\right]<0.$$

ii) the **Lax entropy condition** is satisfied

$$\lambda_L(U_{L*})<\sigma_L<\lambda_L(U_L)\qquad\text{and}\qquad\lambda_R(U_R)<\sigma_R<\lambda_R(U_{R*}).$$

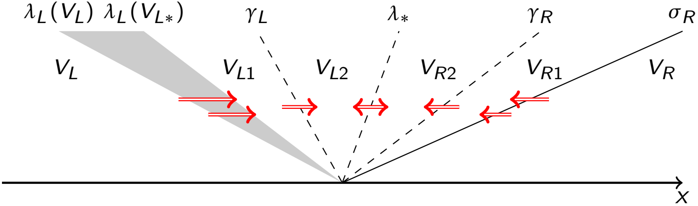
Remark : The transverse kinetic energy is **preserved if there is no-coalescence** and **dissipated if there is coalescence**. More precisely we have

$$\left[\frac{\overline{v}^2+\widehat{v}^2}{2}h\overline{w}+h\overline{v}\widehat{v}\widehat{w}\right]=\frac{1}{2Q}\left[h^2\left(\overline{w}^2-\widehat{w}^2\right)\widehat{v}^2\right].$$

Theorem : Riemann problem of $(2D - SW_2)$

Consider the initial condition $V(0,x) = \begin{cases} V_L = (h_L, \overline{u}_L, \widehat{u}_L, \overline{v}_L, \widehat{v}_L)^t \in \mathbb{R}_+^* \times \mathbb{R}^4 & \text{if } x < 0, \\ V_R = (h_R, \overline{u}_R, \widehat{u}_R, \overline{v}_R, \widehat{v}_R)^t \in \mathbb{R}_+^* \times \mathbb{R}^4 & \text{if } x \geq 0. \end{cases}$

If the following condition is fulfilled : $\overline{u}_R - \overline{u}_L < \mu^r(V_L) + \mu^r(V_R)$
then there exists a **unique selfsimilar** solution $V \in (L^\infty(\mathbb{R}_+^* \times \mathbb{R}))^5$ to the 2D Riemann problem (SW_2) satisfying the **mechanical energy dissipation**.
In addition the water depth h is **strictly positive** for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.



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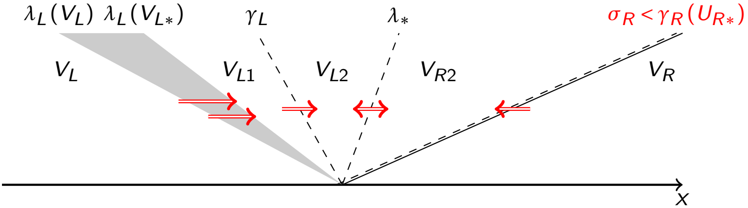
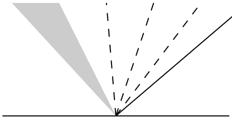
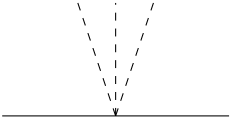
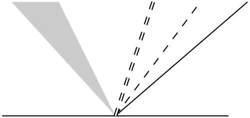




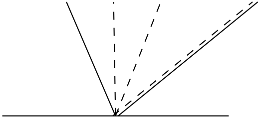
general



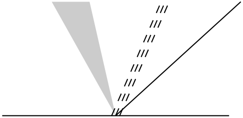
circulation



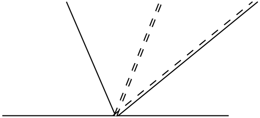
simple resonance



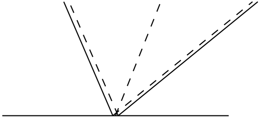
coalescence



double resonance



coalescence + resonance



2 coalescences

Realizations:

- well-possedness of the bi-layerswise discretized model ($2D-SW$).
- analysis of resonance phenomena, coalescence phenomena, non-conservative products

[Aguillon, Audusse, Godlewski, Parisot] to appear...

Perspectives for analysis:

- with an arbitrary number of layers **out of reach**
- with 3 layers (Skewness), 4 layers (Kurtosis)...
- in an asymptotic regime : small shear $\frac{|u_{i+1}-u_i|}{\sqrt{gh}} \ll 1$ **oceanography**
- with active pollutant **oceanography**

Perspectives for numeric:

- preservation of the steady state with circulation
- numerical analysis in case of coalescence
- explanation of the uncoupled numerical scheme CFL **not enough** restrictive