# On the road to Navier-Stokes

**Yohan Penel**

Team ANGE (CEREMA, Inria, UPMC, CNRS)

**Project leader: Cindy Guichard (UPMC)**

COmplex Rheology SURface Flows

Cargèse – June, 1st. 2017

# Outline

# Description

**Project leader**: Cindy Guichard

**Funding**: 8000€

**Members**:

- Marie-Odile Bristeau (Inria Paris)
- Enrique Fernández-Nieto (Spain, Univ. Sevilla)
- Anne Mangeney (IPGP)
- Bernard di Martino (Univ. Corse, ANGE)
- Martin Parisot (Inria)
- Yohan Penel (CEREMA)
- Jacques Sainte-Marie (CEREMA)

# Achievements

**Workshop**:

- *Complex rheology of granular flows: barriers, challenges and deadlocks*
- October, 13th-14th 2016
- 30 participants, 4 speakers (C. Ancey, E. Lemaire, G. Ovarlez, J. Weiss)

**Articles**:

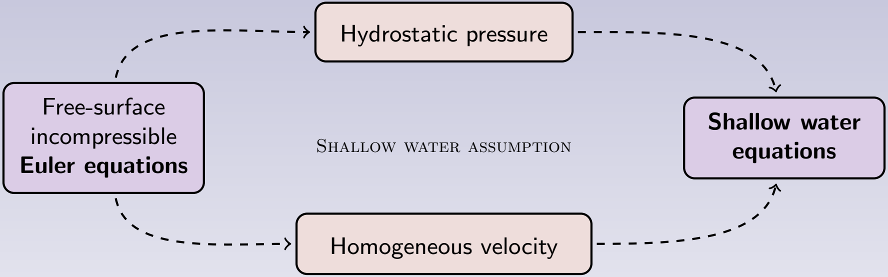
M.-O. Bristeau, C. Guichard, B. di Martino, J. Sainte-Marie, *Layer-averaged Euler and Navier-Stokes equations* (**Comm. Math. Sci.**, to appear)

E. Fernández-Nieto, M. Parisot, Y. Penel, J. Sainte-Marie, *A hierarchy of non-hydrostatic layer-averaged approximations of Euler equations for free surface flows* (submitted)

# Literature about free-surface flows

Free-surface incompressible **Euler equations**

# Literature about free-surface flows



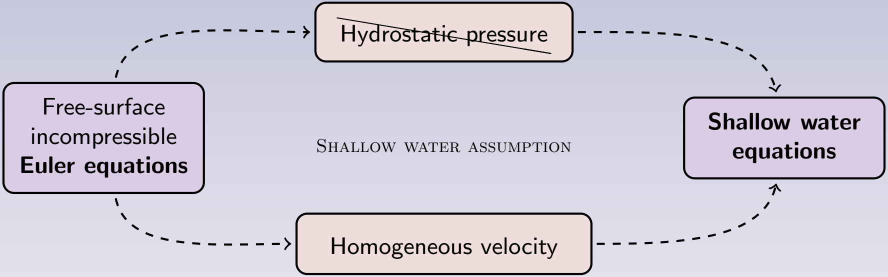

A. Barré de Saint-Venant, *Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leurs lits* (**C. R. Acad. Sci.** 73, 1871)

J.-F. Gerbeau, B. Perthame, *Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation* (**Discrete Contin. Dyn. Syst. Ser. B** 1(1), 2001)

S. Ferrari, F. Saleri, *A new two-dimensional Shallow Water model including pressure effects and slow varying bottom topography* (**Math. Model. Numer. Anal.** 38(2), 2004)

F. Marche, *Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects* (**Eur. J. Mech. B Fluids** 26(1), 2007)

# Literature about free-surface flows




F. Serre, *Contribution à l'étude des écoulements permanents et variables dans les canaux* (**La Houille Blanche** 6, 1953)

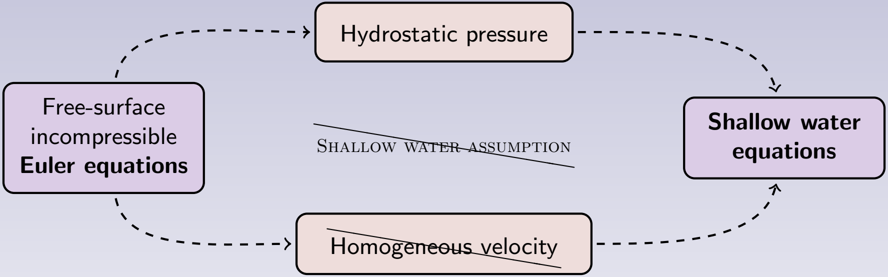
A.E. Green, P.M. Naghdi, *A derivation of equations for wave propagation in water of variable depth* (**J. Fluid Mech.** 78(2), 1976)

M.-O. Bristeau, J. Sainte-Marie, *Derivation of a non-hydrostatic shallow water model; Comparison with Saint-Venant and Boussinesq systems* (**Discrete Contin. Dyn. Syst. Ser. B** 10(4), 2008)

D. Lannes, P. Bonneton, *Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation* (**Phys. Fluids** 21(1), 2009)

Peregrine '67, Madsen *et al.* '91 '96 '03 '06, Nwogu '93, Casulli *et al.* '95 '99, Yamazaki *et al.* '09, . . .

# Literature about free-surface flows



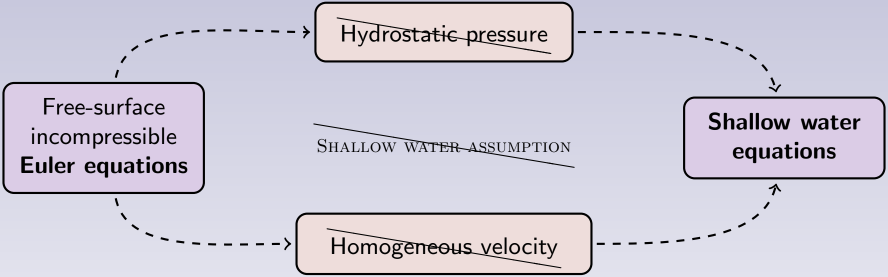

E. Audusse, M.-O. Bristeau, B. Perthame, J. Sainte-Marie, *A multilayer Saint-Venant system with mass exchanges for Shallow Water flows. Derivation and numerical validation* (**Math. Model. Numer. Anal.** 45(1), 2011)

F. Bouchut, V. Zeitlin, *A robust well-balanced scheme for multi-layer shallow water equations* (**Discrete Contin. Dyn. Syst. Ser. B** 13(4), 2010)

E.D. Fernández-Nieto, E.H. Koné, T. Morales de Luna, R. Bürger, *A multilayer shallow water system for polydisperse sedimentation* (**J. Comput. Phys.** 238, 2013)

Castro *et al.* '01 '04 '10, Narbona *et al.* '09 '13, . . .

# Literature about free-surface flows

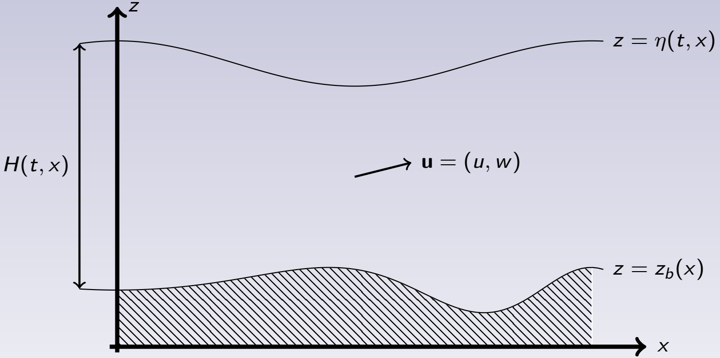



## Derivation of multilayer non-hydrostatic models

M. Zijlema, G.S. Stelling, *Further experiences with computing non-hydrostatic free-surface flows involving water waves* (**Int. J. Numer. Methods Fluids** 48(2), 2005)

Y. Bai, K.F. Cheung, *Dispersion and nonlinearity of multi-layer non-hydrostatic free-surface flow* (**J. Fluid Mech.** 726, 2013)

# Fluid domain




Water height:

$$H(t,x) = \eta(t,x) - z_b(x)$$

# Euler equations

## Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x(u^2 + p) + \partial_z(uw) = 0 \\ \partial_t w + \partial_x(uw) + \partial_z(w^2 + p) = -g \end{cases}$$

set in the domain $\Omega(t) = \left\{(x,z) \in \mathbb{R}^2 \;\middle|\; z_b(x) \leq z \leq \eta(t,x)\right\}$

## Boundary conditions

$$\partial_t \eta(t,x) + u\big(t,x,\eta(t,x)\big)\partial_x \eta(t,x) - w\big(t,x,\eta(t,x)\big) = 0$$
$$p\big(t,x,\eta(t,x)\big) = p^{atm}(t,x)$$
$$u\big(t,x,z_b(x)\big)z_b'(x) - w\big(t,x,z_b(x)\big) = 0$$

together with well-prepared initial conditions

**Pressure fields** $p(t,x,z) = p^{atm}(t,x) + g\big(\eta(t,x) - z\big) + q(t,x,z)$

# Euler equations

## Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x (u^2 + q) + \partial_z (uw) = -\partial_x (g\eta + p^{atm}) \\ \partial_t w + \partial_x (uw) + \partial_z (w^2 + q) = 0 \end{cases}$$

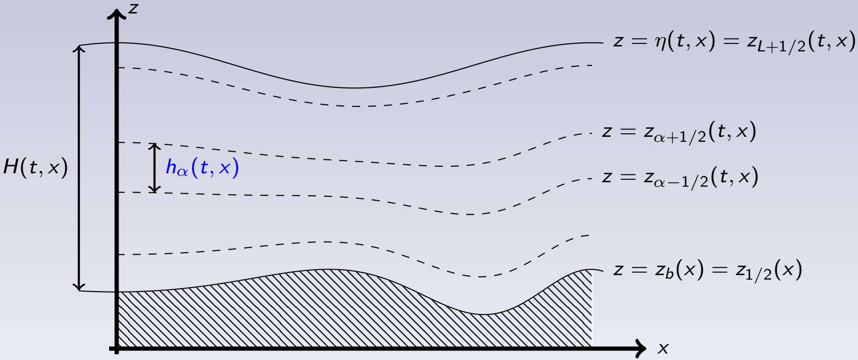
set in the domain $\Omega(t) = \left\{ (x,z) \in \mathbb{R}^2 \;\middle|\; z_b(x) \leq z \leq \eta(t,x) \right\}$

## Boundary conditions

$$\partial_t \eta(t,x) + u\big(t,x,\eta(t,x)\big)\partial_x \eta(t,x) - w\big(t,x,\eta(t,x)\big) = 0$$
$$q\big(t,x,\eta(t,x)\big) = 0$$
$$u\big(t,x,z_b(x)\big) z_b'(x) - w\big(t,x,z_b(x)\big) = 0$$

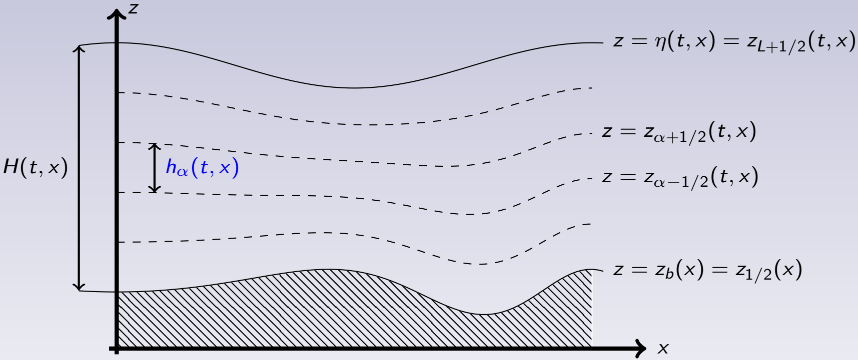
together with well-prepared initial conditions

# Multilayer framework




Height decomposition: $h_\alpha(t,x) = \ell_\alpha H(t,x)$ with $\ell_\alpha \in (0,1)$ and $\sum_{\alpha=1}^{L} \ell_\alpha = 1$

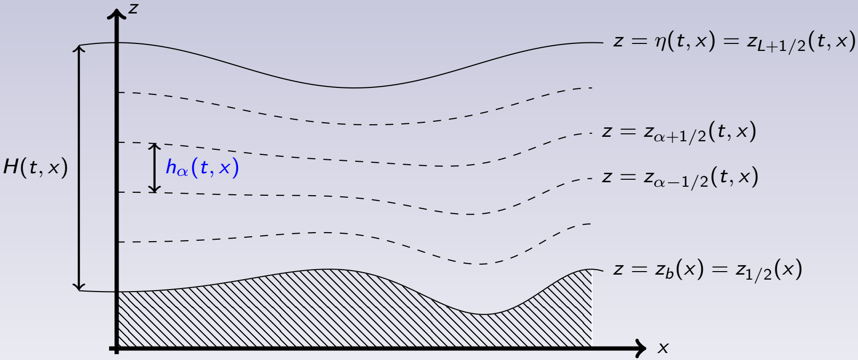
# Multilayer framework




Height decomposition: $h_\alpha(t,x) = \ell_\alpha H(t,x)$ with $\ell_\alpha \in (0,1)$ and $\sum_{\alpha=1}^{L} \ell_\alpha = 1$

**Homogeneous mesh:** $\ell_\alpha = \dfrac{1}{L}$

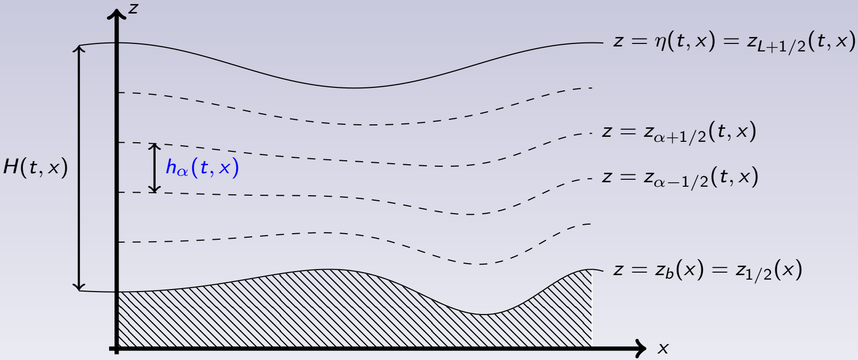
# Multilayer framework




## Notations

$$[\![f]\!]_{\alpha+1/2} = f^+_{\alpha+1/2} - f^-_{\alpha+1/2}, \quad \widetilde{f}_{\alpha+1/2} = \gamma_{\alpha+1/2} f^-_{\alpha+1/2} + (1-\gamma_{\alpha+1/2}) f^+_{\alpha+1/2}$$

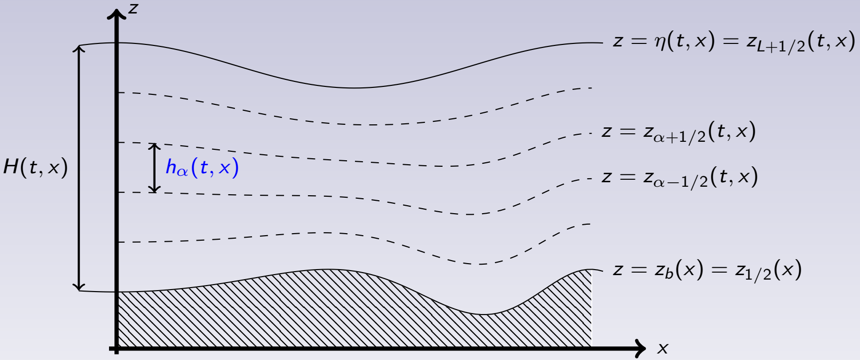
# Multilayer framework




## Notations

$$\langle f \rangle_\alpha(t,x) = \frac{1}{h_\alpha(t,x)} \int_{z_{\alpha-1/2}(t,x)}^{z_{\alpha+1/2}(t,x)} f(t,x,z)\,\mathrm{d}z$$

# Multilayer framework




## Notations

$$\mathbf{n}_{\alpha+1/2} = (-\partial_x z_{\alpha+1/2}, 1)^T$$

# Discontinuous Galerkin framework

Let us be given a velocity field satisfying

$$[\![\mathbf{u}]\!]_{\alpha+1/2} \cdot \mathbf{n}_{\alpha+1/2} = 0 \iff [\![w]\!]_{\alpha+1/2} = [\![u]\!]_{\alpha+1/2} \partial_x z_{\alpha+1/2}$$

**Toy model**

$$\partial_t \mathscr{R} + \partial_x (u\mathscr{R} + \mathscr{P}) + \partial_z (w\mathscr{R} + \mathscr{Q}) = \mathscr{S} \qquad (1)$$

where $\mathscr{R}$, $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{S}$ take values in $\mathbb{R}^p$

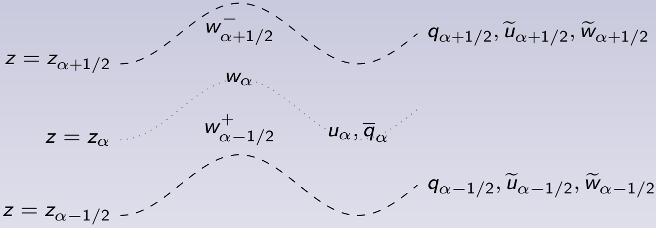
**Semi-discrete formulation** over each layer $\mathcal{L}_\alpha = (z_{\alpha+1/2}, z_{\alpha-1/2})$

$$\partial_t (h_\alpha \langle \mathscr{R} \rangle_\alpha) + \partial_x (h_\alpha [\langle u\mathscr{R} \rangle_\alpha + \langle \mathscr{P} \rangle_\alpha]) + \mathscr{F}^{\mathscr{R}}_{\alpha+1/2} - \mathscr{F}^{\mathscr{R}}_{\alpha-1/2} = h_\alpha \langle \mathscr{S} \rangle_\alpha$$

where

$$\mathscr{F}^{\mathscr{R}}_{\alpha+1/2} = \Gamma_{\alpha+1/2} \widetilde{\mathscr{R}}_{\alpha+1/2} - \widetilde{\mathscr{P}}_{\alpha+1/2} \partial_x z_{\alpha+1/2} + \widetilde{\mathscr{Q}}_{\alpha+1/2}$$
$$\Gamma_{\alpha+1/2} = \widetilde{w}_{\alpha-1/2} \partial_t z_{\alpha+1/2} - \widetilde{u}_{\alpha+1/2} \partial_x z_{\alpha+1/2}$$

# Spaces of approximation




$$u(t,x) = \sum_{\alpha=1}^{L} u_\alpha(t,x) \mathbb{1}_{\{\mathcal{L}_\alpha(t,x)\}}(z) + \mathcal{E}_L$$

$$w(t,x) = \sum_{\alpha=1}^{L} \left[ w_\alpha(t,x) - \big(z - z_\alpha(t,x)\big)\partial_x u_\alpha(t,x) \right] \mathbb{1}_{\{\mathcal{L}_\alpha(t,x)\}}(z) + \mathcal{E}'_L$$

$q$ continuous over the water column

# Core of the models

Applying the previous semi-discretisation to the Euler equations leads to

$$\begin{cases} \partial_t h_\alpha + \partial_x(h_\alpha u_\alpha) + \Gamma_{\alpha+1/2} - \Gamma_{\alpha-1/2} = 0 \\ \partial_t(h_\alpha u_\alpha) + \partial_x\left(h_\alpha u_\alpha^2 + h_\alpha \overline{q}_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x(g\eta + p^{atm}) \\ \partial_t(h_\alpha w_\alpha) + \partial_x\left(h_\alpha u_\alpha w_\alpha\right) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0 \end{cases}$$

with

$$\begin{cases} \mathcal{U}_{\alpha+1/2} = \widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2} - \partial_x z_{\alpha+1/2} q_{\alpha+1/2} \\ \mathcal{W}_{\alpha+1/2} = \widetilde{w}_{\alpha+1/2}\Gamma_{\alpha+1/2} + q_{\alpha+1/2} \end{cases}$$

and

$$\Gamma_{\alpha+1/2} = \widetilde{w}_{\alpha+1/2} - \partial_t z_{\alpha+1/2} - \widetilde{u}_{\alpha+1/2}\partial_x z_{\alpha+1/2}$$

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and

$$\Gamma_{\alpha+1/2} = \sum_{\beta=\alpha+1}^{L} \partial_x\left(h_\beta\left[\overline{u}_\beta - \overline{u}\right]\right) \quad \text{where} \quad \overline{u} = \sum_{\alpha=1}^{L} \ell_\alpha u_\alpha$$

# Core of the models

Applying the previous semi-discretisation to the Euler equations leads to

$$\begin{cases} \partial_t H + \partial_x (H\overline{u}) = 0 \\ \partial_t(h_\alpha u_\alpha) + \partial_x \left(h_\alpha u_\alpha^2 + h_\alpha \overline{q}_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x(g\eta + p^{atm}) \\ \partial_t(h_\alpha w_\alpha) + \partial_x (h_\alpha u_\alpha w_\alpha) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0 \end{cases}$$

with

$$\begin{cases} \mathcal{U}_{\alpha+1/2} = \widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2} - \partial_x z_{\alpha+1/2} q_{\alpha+1/2} \\ \mathcal{W}_{\alpha+1/2} = \widetilde{w}_{\alpha+1/2}\Gamma_{\alpha+1/2} + q_{\alpha+1/2} \end{cases}$$

and

$$\Gamma_{\alpha+1/2} = \sum_{\beta=\alpha+1}^{L} \partial_x \left(h_\beta \left[\overline{u}_\beta - \overline{u}\right]\right) \quad \text{where} \quad \overline{u} = \sum_{\alpha=1}^{L} \ell_\alpha u_\alpha$$

# Requirements for the $\mathbb{P}_1$ choice

## Additional equation

$$\partial_t(zw) + \partial_x(zuw) + \partial_z\big(z(w^2+q)\big) = w^2 + q.$$

which is discretised as

$$\partial_t(h_\alpha\langle zw\rangle_\alpha) + \partial_x(h_\alpha u_\alpha\langle zw\rangle_\alpha) + \mathscr{F}^{zw}_{\alpha+1/2} - \mathscr{F}^{zw}_{\alpha-1/2} = h_\alpha\big\langle w^2+q\big\rangle_\alpha$$

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Let us introduce the signed standard deviation $\sigma_\alpha = -\frac{h_\alpha \partial_x u_\alpha}{2\sqrt{3}}$ such that

$$\big\langle w^2\big\rangle_\alpha = w_\alpha^2 + \sigma_\alpha^2, \qquad \langle zw\rangle_\alpha = z_\alpha w_\alpha + \frac{h_\alpha \sigma_\alpha}{2\sqrt{3}}$$

# Requirements for the $\mathbb{P}_1$ choice

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Then

$$\partial_t(h_\alpha z_\alpha w_\alpha) + \partial_x(h_\alpha z_\alpha u_\alpha w_\alpha) + \partial_t\left(\frac{h_\alpha^2\sigma_\alpha}{2\sqrt{3}}\right) + \partial_x\left(\frac{h_\alpha^2\sigma_\alpha u_\alpha}{2\sqrt{3}}\right)$$
$$+ z_{\alpha+1/2}(\widetilde{w}_{\alpha+1/2}\Gamma_{\alpha+1/2} + q_{\alpha+1/2}) - z_{\alpha-1/2}(\widetilde{w}_{\alpha-1/2}\Gamma_{\alpha-1/2} + q_{\alpha-1/2})$$
$$= h_\alpha\left(w_\alpha^2 + \sigma_\alpha^2 + \overline{q}_\alpha\right)$$

# Requirements for the $\mathbb{P}_1$ choice

## Additional equation

$$\partial_t(zw) + \partial_x(zuw) + \partial_z\big(z(w^2+q)\big) = w^2 + q.$$

which is discretised as

$$\partial_t(h_\alpha \langle zw \rangle_\alpha) + \partial_x(h_\alpha u_\alpha \langle zw \rangle_\alpha) + \mathscr{F}^{zw}_{\alpha+1/2} - \mathscr{F}^{zw}_{\alpha-1/2} = h_\alpha \big\langle w^2 + q \big\rangle_\alpha$$

Then

$$\partial_t(h_\alpha \sigma_\alpha) + \partial_x(h_\alpha \sigma_\alpha u_\alpha) = 2\sqrt{3}\left[\overline{q}_\alpha - \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} - \Gamma_{\alpha+1/2}\left(\frac{h_\alpha \partial_x u_\alpha}{12} + \frac{\widetilde{w}_{\alpha+1/2} - w_\alpha}{2}\right) + \Gamma_{\alpha-1/2}\left(\frac{h_\alpha \partial_x u_\alpha}{12} + \frac{w_\alpha - \widetilde{w}_{\alpha-1/2}}{2}\right)\right]$$

# Requirements for the $\mathbb{P}_1$ choice

## Additional equation

$$\partial_t(zw) + \partial_x(zuw) + \partial_z\big(z(w^2+q)\big) = w^2 + q.$$

which is discretised as

$$\partial_t(h_\alpha \langle zw\rangle_\alpha) + \partial_x(h_\alpha u_\alpha \langle zw\rangle_\alpha) + \mathscr{F}^{zw}_{\alpha+1/2} - \mathscr{F}^{zw}_{\alpha-1/2} = h_\alpha \big\langle w^2 + q\big\rangle_\alpha$$

Rather using a Hermite interpolation leads to

$$zw_{|\mathcal{L}_\alpha} \approx z_\alpha w_\alpha + (z - z_\alpha)(w_\alpha - z_\alpha \partial_x u_\alpha), \qquad w^2_{|\mathcal{L}_\alpha} \approx w_\alpha^2 - 2(z - z_\alpha) w_\alpha \partial_x u_\alpha$$

# Requirements for the $\mathbb{P}_1$ choice

## Additional equation

$$\partial_t(zw) + \partial_x(zuw) + \partial_z\big(z(w^2+q)\big) = w^2 + q.$$

which is discretised as

$$\partial_t\big(h_\alpha\langle zw\rangle_\alpha\big) + \partial_x\big(h_\alpha u_\alpha \langle zw\rangle_\alpha\big) + \mathscr{F}^{zw}_{\alpha+1/2} - \mathscr{F}^{zw}_{\alpha-1/2} = h_\alpha\big\langle w^2+q\big\rangle_\alpha$$

Then

$$\begin{aligned}\partial_t(h_\alpha z_\alpha w_\alpha) + \partial_x(h_\alpha z_\alpha u_\alpha w_\alpha) + z_{\alpha+1/2}q_{\alpha+1/2} - z_{\alpha-1/2}q_{\alpha-1/2}\\ + \Gamma_{\alpha+1/2}\left(z_{\alpha+1/2}\widetilde{w}_{\alpha+1/2} + \frac{H^2}{4L^2}\big(\widetilde{\partial_x\overline{u}}\big)_{\alpha+1/2}\right)\\ - \Gamma_{\alpha-1/2}\left(z_{\alpha-1/2}\widetilde{w}_{\alpha-1/2} + \frac{H^2}{4L^2}\big(\widetilde{\partial_x\overline{u}}\big)_{\alpha-1/2}\right) = h_\alpha\left(w_\alpha^2 + \overline{q}_\alpha\right)\end{aligned}$$

# Requirements for the $\mathbb{P}_1$ choice

## Additional equation

$$\partial_t(zw) + \partial_x(zuw) + \partial_z\big(z(w^2 + q)\big) = w^2 + q.$$

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Then

$$\begin{aligned}\overline{q}_\alpha = \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} &+ \Gamma_{\alpha+1/2}\left(\frac{H}{4L}\widetilde{(\partial_x \overline{u})}_{\alpha+1/2} + \frac{\widetilde{w}_{\alpha+1/2} - w_\alpha}{2}\right)\\ &- \Gamma_{\alpha-1/2}\left(\frac{H}{4L}\widetilde{(\partial_x \overline{u})}_{\alpha-1/2} + \frac{w_\alpha - \widetilde{w}_{\alpha-1/2}}{2}\right)\end{aligned}$$

# $\mathrm{NHML}_2$ model: $u_{|\mathcal{L}_\alpha} \in \mathbb{P}_0, w_{|\mathcal{L}_\alpha} \in \mathbb{P}_1, q_{|\mathcal{L}_\alpha} \in \mathbb{P}_2, E_{|\mathcal{L}_\alpha} \in \mathbb{P}_2$

$$\partial_t H + \partial_x (H\overline{u}) = 0$$

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$$\partial_t(h_\alpha u_\alpha) + \partial_x \left(h_\alpha u_\alpha^2 + h_\alpha \overline{q}_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x(g\eta + p^{atm})$$

$$\partial_t(h_\alpha w_\alpha) + \partial_x (h_\alpha u_\alpha w_\alpha) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0$$

$$\partial_t(h_\alpha \sigma_\alpha) + \partial_x(h_\alpha \sigma_\alpha u_\alpha) = 2\sqrt{3}\left[\overline{q}_\alpha - \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} - \Gamma_{\alpha+1/2}\left(\frac{h_\alpha \partial_x u_\alpha}{12} + \frac{\widetilde{w}_{\alpha+1/2} - w_\alpha}{2}\right) + \Gamma_{\alpha-1/2}\left(\frac{h_\alpha \partial_x u_\alpha}{12} + \frac{w_\alpha - \widetilde{w}_{\alpha-1/2}}{2}\right)\right]$$

---

$$\partial_x u_\alpha + \frac{w^-_{\alpha+1/2} - w_\alpha}{h_\alpha/2} = 0 \qquad \sigma_\alpha = -\frac{h_\alpha \partial_x u_\alpha}{2\sqrt{3}}$$

$$w^+_{\alpha-1/2} - u_\alpha \partial_x z_{\alpha-1/2} + \sum_{\beta=1}^{\alpha-1} \partial_x(h_\beta \overline{u}_\beta) = 0$$

# $\mathrm{NHML}_1$ model: $u_{|\mathcal{L}_\alpha} \in \mathbb{P}_0, w_{|\mathcal{L}_\alpha} \in \mathbb{P}_1, q_{|\mathcal{L}_\alpha} \in \mathbb{P}_2, E_{|\mathcal{L}_\alpha} \in \mathbb{P}_0$

$$\partial_t H + \partial_x (H\overline{u}) = 0$$

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$$\partial_t(h_\alpha u_\alpha) + \partial_x \left(h_\alpha u_\alpha^2 + h_\alpha \overline{q}_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x(g\eta + p^{atm})$$

$$\partial_t(h_\alpha w_\alpha) + \partial_x (h_\alpha u_\alpha w_\alpha) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0$$

$$\overline{q}_\alpha = \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} + \Gamma_{\alpha+1/2}\left(\frac{H}{4L}\widetilde{(\partial_x \overline{u})}_{\alpha+1/2} + \frac{\widetilde{w}_{\alpha+1/2} - w_\alpha}{2}\right) - \Gamma_{\alpha-1/2}\left(\frac{H}{4L}\widetilde{(\partial_x \overline{u})}_{\alpha-1/2} + \frac{w_\alpha - \widetilde{w}_{\alpha-1/2}}{2}\right)$$

---

$$\partial_x u_\alpha + \frac{w^-_{\alpha+1/2} - w_\alpha}{h_\alpha/2} = 0 \qquad \sigma_\alpha = -\frac{h_\alpha \partial_x u_\alpha}{2\sqrt{3}}$$

$$w^+_{\alpha-1/2} - u_\alpha \partial_x z_{\alpha-1/2} + \sum_{\beta=1}^{\alpha-1} \partial_x(h_\beta \overline{u}_\beta) = 0$$

# $\mathrm{NHML}_0$ model: $u_{|\mathcal{L}_\alpha} \in \mathbb{P}_0, w_{|\mathcal{L}_\alpha} \in \mathbb{P}_0, q_{|\mathcal{L}_\alpha} \in \mathbb{P}_1, E_{|\mathcal{L}_\alpha} \in \mathbb{P}_0$

$$\partial_t H + \partial_x (H\overline{u}) = 0$$

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$$\partial_t(h_\alpha u_\alpha) + \partial_x \left(h_\alpha u_\alpha^2 + h_\alpha \overline{q}_\alpha\right) + \mathcal{U}_{\alpha+1/2} - \mathcal{U}_{\alpha-1/2} = -h_\alpha \partial_x(g\eta + p^{atm})$$

$$\partial_t(h_\alpha w_\alpha) + \partial_x (h_\alpha u_\alpha w_\alpha) + \mathcal{W}_{\alpha+1/2} - \mathcal{W}_{\alpha-1/2} = 0$$

$$\overline{q}_\alpha = \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2}$$

---

$$w_\alpha - u_\alpha \partial_x z_\alpha + \sum_{\beta=1}^{\alpha-1} \partial_x(h_\beta \overline{u}_\beta) + \frac{1}{2}\partial_x(h_\alpha u_\alpha) = 0$$

# Energy inequality

Denoting $\mathcal{K} = \frac{u^2+w^2}{2}$, smooth solutions to the Euler equations satisfy

$$\partial_t \left( \int_{z_b}^{\eta} \left( \mathcal{K} + g\frac{\eta + z_b}{2} + p^{atm} \right) \mathrm{d}z \right) + \partial_x \left( \int_{z_b}^{\eta} u(\mathcal{K} + q + g\eta + p^{atm})\, \mathrm{d}z \right) = H\partial_t p^{atm}.$$

## Proposition

*Let us assume that* $\left(\gamma_{\alpha+1/2} - \frac{1}{2}\right)\Gamma_{\alpha+1/2} \geq 0$. *If* $(H, u_\alpha, w_\alpha, \overline{q}_\alpha)$ *are smooth solutions to the* $\mathrm{NHML}_2$*-model, then with* $\overline{\mathcal{K}}_\alpha = \frac{u_\alpha^2+w_\alpha^2+\sigma_\alpha^2}{2}$

$$\partial_t \left[ \sum_{\alpha=1}^{L} h_\alpha \left( \overline{\mathcal{K}}_\alpha + g z_\alpha + p^{atm} \right) \right] + \partial_x \left[ \sum_{\alpha=1}^{L} h_\alpha u_\alpha \left( \overline{\mathcal{K}}_\alpha + \overline{q}_\alpha + g\eta + p^{atm} \right) \right] \leq H\partial_t p^{atm}.$$

# Comment

Constraint $\left(\gamma_{\alpha+1/2} - \frac{1}{2}\right)\Gamma_{\alpha+1/2} \geq 0$ is equivalent to taking

$$\gamma_{\alpha+1/2} = \frac{1}{2}\left(1 + \lambda\,\mathrm{sign}(\Gamma_{\alpha+1/2})\right)$$

for any $\lambda \geq 0$, which gives

$$\widetilde{\mathscr{R}}_{\alpha+1/2}\Gamma_{\alpha+1/2} = \frac{\mathscr{R}^{+}_{\alpha+1/2} + \mathscr{R}^{-}_{\alpha+1/2}}{2}\Gamma_{\alpha+1/2} - \frac{\lambda}{2}|\Gamma_{\alpha+1/2}|\left(\mathscr{R}^{+}_{\alpha+1/2} - \mathscr{R}^{-}_{\alpha+1/2}\right).$$

The energy inequality is satisfied in particular for $\gamma_{\alpha+1/2} = \frac{1}{2}$ ($\lambda = 0$) and for $\gamma_{\alpha+1/2} = \mathbb{1}_{\{\Gamma_{\alpha+1/2} \geq 0\}}$ ($\lambda = 1$).

# Linear dispersion relation

Let us linearise around the so-called lake-at-rest steady state $(H_0, 0, 0, 0)$.

## Proposition

*There exists a plane wave solution* $\left(\hat{H}, \hat{u}_\alpha, \hat{w}_\alpha, \hat{q}_\alpha\right) e^{i(kx-\omega t)}$ *to the linearised* $\mathrm{NHML}_k$ *system provided the following dispersion relation holds*

$$c_L^2(kH_0) = \frac{\omega^2}{k^2 g H_0} = \frac{\mathcal{P}_L(kH_0)}{\mathcal{Q}_L(kH_0)}$$

*where* $\mathcal{P}_L$ *and* $\mathcal{Q}_L$ *are explicit polynomials. Moreover when the number of layers* $L$ *goes to infinity,* $c_L^2$ *tends to*

$$c_{Airy}^2(kH_0) = \frac{\tanh(kH_0)}{kH_0}.$$

# $\mathrm{NHML}_2$ for $L \in \{1, 2, 3\}$

| $L$ | $\mathcal{P}_L$ | $\mathcal{Q}_L$ |
|---|---|---|
| 1 | $1$ | $1 + \frac{x^2}{3}$ |
| 2 | $1 + \frac{x^2}{12}$ | $1 + \frac{5x^2}{12} + \frac{7x^4}{576}$ |
| 3 | $1 + \frac{x^2}{9} + \frac{5x^4}{2916}$ | $1 + \frac{4x^2}{9} + \frac{19x^4}{972} + \frac{13x^6}{78732}$ |

# $\mathrm{NHML}_1$ for $L \in \{1, 2, 3\}$

| $L$ | $\mathcal{P}_L$ | $\mathcal{Q}_L$ |
|---|---|---|
| 1 | $1$ | $1 + \frac{x^2}{4}$ |
| 2 | $1 + \frac{x^2}{16}$ | $1 + \frac{3x^2}{8} + \frac{x^4}{256}$ |
| 3 | $1 + \frac{5x^2}{54} + \frac{x^4}{1296}$ | $1 + \frac{5x^2}{12} + \frac{5x^4}{432} + \frac{x^6}{46656}$ |

# $L \geq 4$ ($\lambda = 3$ for $\mathrm{NHML}_2$ and $\lambda = 2$ for $\mathrm{NHML}_{1/0}$)

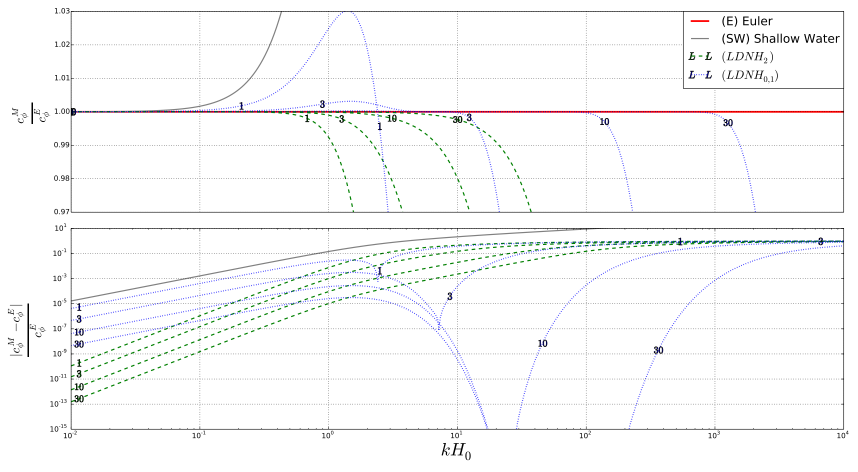
$$\mathcal{P}_L(x) = \frac{1}{L}\left[\left(1 - \frac{x^2}{2\lambda L^2}\right)^{L-1} + \xi_{L-4}\left(1 - \frac{x^2}{2\lambda L^2}\right)^2 - \xi_{L-3}\left(1 + \frac{2\lambda - 1}{2\lambda}\frac{x^2}{L^2}\right)\right]$$

$$\mathcal{Q}_L(x) = \left(1 - \frac{x^2}{2\lambda L^2}\right)^{L-1}\left(1 + \frac{\lambda - 1}{2\lambda}\frac{x^2}{L^2}\right) + \left(1 - \frac{x^2}{2\lambda L^2}\right)^2 \frac{x^2 \xi_{L-4}}{2L^2} - \left(3 + \frac{2\lambda - 3}{2\lambda}\frac{x^2}{L^2}\right)\frac{x^2\xi_{L-3}}{2L^2}$$

$$\xi_k = \frac{L^2}{x^2}\left(1 - \frac{x^2}{2\lambda L^2}\right)^{k+2} + \Xi_e \sum_{0 \leq 2m \leq k} \binom{k}{2m}\left(1 + \frac{\lambda - 1}{2\lambda}\frac{x^2}{L^2}\right)^{k-2m}\frac{x^{2m-1}}{L^{2m-1}}\left(1 + \frac{\lambda - 2}{4\lambda}\frac{x^2}{L^2}\right)^m + \Xi_o \sum_{0 \leq 2m+1 \leq k} \binom{k}{2m+1}\left(1 + \frac{\lambda - 1}{2\lambda}\frac{x^2}{L^2}\right)^{k-2m-1}\frac{x^{2m+1}}{L^{2m+1}}\left(1 + \frac{\lambda - 2}{4\lambda}\frac{x^2}{L^2}\right)^m$$

where $\Xi_e = -1 + \frac{1-3\lambda}{\lambda}\frac{x^2}{L^2} + \frac{-1+6\lambda-4\lambda^2}{4\lambda^2}\frac{x^4}{L^4}$ and $\Xi_o = -\frac{5}{2} + \frac{5(1-\lambda)}{2\lambda}\frac{x^2}{L^2} + \frac{-\frac{5}{2}+5\lambda-2\lambda^2}{4\lambda^2}\frac{x^4}{L^4}$.

# Phase velocity



# Navier-Stokes for a hydrostatic flow

## Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x(u^2) + \partial_z(uw) + \partial_x p = \partial_x \Sigma_{xx} + \partial_z \Sigma_{xz} \\ \partial_z p = -g + \partial_x \Sigma_{zx} + \partial_z \Sigma_{zz} \end{cases}$$

# Navier-Stokes for a hydrostatic flow

## Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x(u^2) + \partial_z(uw) + g\partial_x \eta = \partial_x \Sigma_{xx} + \partial_z \Sigma_{xz} + \partial^2_{xx}\left(\int_z^{\eta(t,x)} \Sigma_{zx}\right) - \partial_x \Sigma_{zz} \\ p = g(\eta(t,x) - z) - \partial_x \left(\int_z^{\eta(t,x)} \Sigma_{zx}\right) + \Sigma_{zz} \end{cases}$$

# Navier-Stokes for a hydrostatic flow

## Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x(u^2) + \partial_z(uw) + g\partial_x \eta = \partial_x \Sigma_{xx} + \partial_z \Sigma_{xz} + \partial^2_{xx}\left(\int_z^{\eta(t,x)} \Sigma_{zx}\right) - \partial_x \Sigma_{zz} \\ p = g(\eta(t,x) - z) - \partial_x \left(\int_z^{\eta(t,x)} \Sigma_{zx}\right) + \Sigma_{zz} \end{cases}$$

## Layer-averaging

$$\begin{cases} \partial_t H + \partial_x(H\overline{u}) = 0 \\ \partial_t(h_\alpha u_\alpha) + \partial_x\left(h_\alpha u_\alpha^2 + g\frac{h_\alpha^2}{2\ell_\alpha}\right) + \widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2} - \widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2} \\ \qquad = -gh_\alpha \partial_x z_b + \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left(\partial_x \Sigma_{xx} + \partial_z \Sigma_{xz} + \partial^2_{xx}\left(\int_z^{\eta(t,x)} \Sigma_{zx}\right) - \partial_x \Sigma_{zz}\right) \end{cases}$$

# Discretisation of viscous terms

$$\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left( \partial_x \Sigma_{xx} - \partial_x \Sigma_{zz} + \partial_{xx}^2 \left( \int_z^{\eta(t,x)} \Sigma_{zx} \right) + \partial_z \Sigma_{xz} \right) \approx$$

$$\begin{aligned}
&\partial_x \big( h_\alpha (\Sigma_{xx,\alpha} - \Sigma_{zz,\alpha}) \big) - \partial_x z_{\alpha+1/2} (\Sigma_{xx,\alpha+1/2} - \Sigma_{zz,\alpha+1/2}) \\
&\quad + \partial_x z_{\alpha-1/2} (\Sigma_{xx,\alpha-1/2} - \Sigma_{zz,\alpha-1/2}) \\
&+ \partial_{xx}^2 (h_\alpha z_\alpha \Sigma_{zx,\alpha}) + z_{\alpha+1/2} \partial_{xx}^2 \left( \sum_{\beta=\alpha+1}^{L} h_\beta \Sigma_{zx,\beta} \right) - \Sigma_{zx,\alpha+1/2} \partial_{xx}^2 z_{\alpha+1/2} \\
&\quad - z_{\alpha-1/2} \partial_{xx}^2 \left( \sum_{\beta=\alpha}^{L} h_\beta \Sigma_{zx,\beta} \right) + \Sigma_{zx,\alpha-1/2} \partial_{xx}^2 z_{\alpha-1/2} \\
&+ \Sigma_{xz,\alpha+1/2} - \Sigma_{xz,\alpha-1/2}
\end{aligned}$$

# Conclusion

- Derivation of a class of multilayer non-hydrostatic models as semi-discretisations of the Euler equations
- Analysis of physical properties (energy, hydrodynamic balances, dispersive effects)
- Study of the discretisation of viscous terms in accordance with energy estimates
- **On-going works**
  - Numerical strategies for each model
  - Incorporation of viscous effects in the non-hydrostatic framework

# Thank you for your attention . . .

. . . and see you in November at Inria Paris

**Workshop:**
**An overview on free surface flows**