A Newton method for viscoplastic flows

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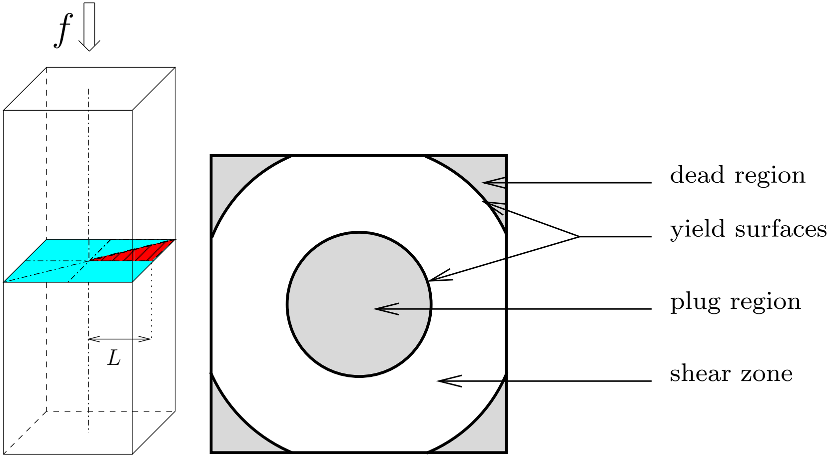
Motivations

- natural hazards : evaluation & prediction
- industry : cements, forming processes of clays & metalic alloy
- biology : blood flows in small vessels, tissues

Mathematical difficulties

- minimization of a non-differentiable energy:
  $\Rightarrow$ regularization or specific optimization approaches
- poor regularity of the stress (only C^0) and velocity (only C^1):
  $\Rightarrow$ mesh adaptation
- non-unicity of the stress
- slow or inaccurate numerical computations

Benchmark: flow in a tube with square section



Problem statement: Herschel-Bulkley model

(P): *find* $\boldsymbol{\sigma}$ *and* u *defined in* Ω *such that*:

$$
\begin{aligned}
\boldsymbol{\sigma} &= K|\nabla u|^{n-1}\nabla u + \sigma_0 \frac{\nabla u}{|\nabla u|} \quad \text{when } \nabla u \neq 0 \\
|\boldsymbol{\sigma}| &\leqslant \sigma_0 \quad \text{when } \nabla u = 0 \\
\operatorname{div} \boldsymbol{\sigma} &= -f \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

Notations:

$$
\begin{aligned}
\nabla u &= \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \\
\operatorname{div} \boldsymbol{\sigma} &= \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} \\
|\boldsymbol{\sigma}| &= \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2}
\end{aligned}
$$

Former method #1: regularization

$(P)_\varepsilon$: *find* $\boldsymbol{\sigma}_\varepsilon$ *and* u_ε *defined in* Ω *such that*:

$$
\begin{aligned}
\boldsymbol{\sigma}_\varepsilon &= K|\nabla u_\varepsilon|^{n-1}\nabla u_\varepsilon + \sigma_0 \frac{\nabla u_\varepsilon}{\left(|\nabla u_\varepsilon|^2 + \varepsilon^2\right)^{1/2}} \\
\operatorname{div} \boldsymbol{\sigma}_\varepsilon &= -f \text{ in } \Omega \\
u_\varepsilon &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

- advantage: easy to implement: non-constant viscosity
- advantage: fast, Newton method is possible and very efficent
- drawback: inaccurate, no more rigid regions...

Former method #2: augmented Lagrangian

$$u = \arg\min_{v} \int_{\Omega} \frac{K}{n+1} |\nabla v|^{n+1} + \sigma_0 |\nabla v| - fv$$

- advantage: acurate prediction of yield surfaces
- drawback: slow, minimization algorithm

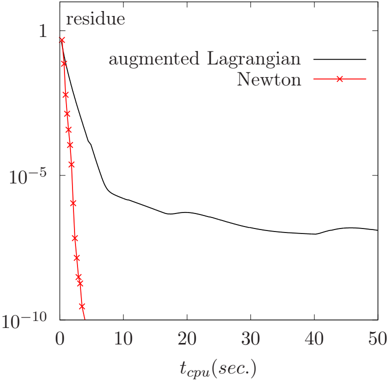
Comparison

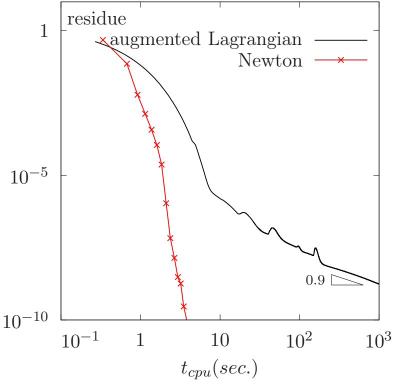
method	advantage	drawback
regularization	fast	inaccurate
AL	accurate	slow

Choices: fast-but-inacurate or slow-and-accurate !

Present contribution: a new fast-and-accurate method

What is fast or slow ?





power-law	$r_n = n^{-\alpha}$	augmented Lagrangian
linear	$r_n = \exp(-\alpha n)$	fixed point ; trust-region
quadratic	$r_n = \alpha\,(r_{n-1})^2$	Newton

History

1969 Hestenes & Powell
augmented Lagrangian methods (AL): small sized problems

1983 Bercovier & Engelman
regularization method: **viscoplastic computations**

1983 Fortin and Glowinski
EDP & AL method (book): *theory, few computations*

1987 Papanastasiou
another regularization method: **viscoplastic computations**

1990-2000 many computations by regularization

History

2001 Roquet & Saramito
mesh adaptive & AL method: **accurate yield surfaces**

2003 Vola, Boscardin & Latché
AL method

2004 Mitouslis & Huilgol
regularization method

2004 Moyers-Gonzalez & Frigaard
compare regularization & AL method

2000-2016 many computations by both AL or regularization

Others attempts : neither regularization nor AL

2010 los Reyes & Gonzáles-Andrades
Semi-smooth Newton method: reduces to regularization

2014 Aposporidis, Vassilevski and Veneziani
Fixed point method: at best, **linear** convergence

2015 Bleyer, Maillard, de Buhan, Coussot
interior point method: reduces to regularization

2015 Treskatis, Moyers-Gonzalez & Price
- accelerated AL method: still **power-law** convergence
- trust-region: at best, **linear** convergence

2016 Chupin and Dubois
Fixed point method: at best, **linear** convergence

Outline

1. Problem reformulation
2. Newton method
3. Results and performances

Reformulation #1

$(Q)_0$: *find* $\boldsymbol{\sigma}$ *and* u *defined in* Ω *such that*:

$$\begin{aligned} \nabla u &= P_0(\boldsymbol{\sigma}) \\ \operatorname{div} \boldsymbol{\sigma} &= -f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

Projector:

$$P_0(\boldsymbol{\tau}) = \begin{cases} K^{-1/n} \left(|\boldsymbol{\tau}| - \sigma_0\right)^{1/n} \dfrac{\boldsymbol{\tau}}{|\boldsymbol{\tau}|} & \text{when } |\boldsymbol{\tau}| > \sigma_0 \\ 0 & \text{otherwise} \end{cases}$$

Reformulation #2

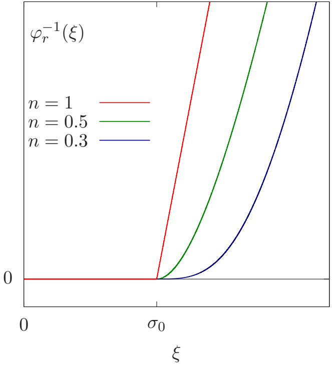
$(Q)_r$: *find* u *and* $\boldsymbol{\beta} = \boldsymbol{\sigma} + r\nabla u$ *such that*

$$\begin{aligned} r\Delta u - \operatorname{div}\boldsymbol{\beta} &= f \text{ in } \Omega \\ \boldsymbol{\nabla} u - P_r(\boldsymbol{\beta}) &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

Extended projector

$$P_r(\boldsymbol{\tau}) = \begin{cases} \varphi_r^{-1}(|\boldsymbol{\tau}|)\dfrac{\boldsymbol{\tau}}{|\boldsymbol{\tau}|} & \text{when } |\boldsymbol{\tau}| > \sigma_0 \\ 0 & \text{otherwise} \end{cases}$$

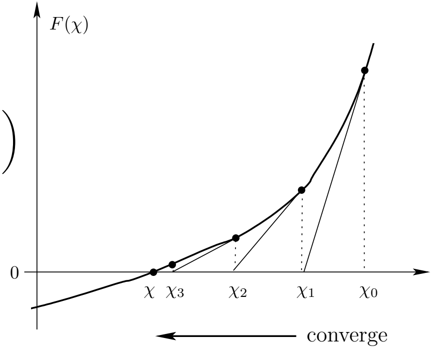
$$\varphi_r(\dot\gamma) = \sigma_0 + K\dot\gamma^n + r\dot\gamma, \quad \forall \dot\gamma \geqslant 0$$



Newton method: $F(\chi) = 0$

$$\chi = (u, \boldsymbol{\beta})$$

$$F(u, \boldsymbol{\beta}) = \begin{pmatrix} r\Delta u - \operatorname{div} \boldsymbol{\beta} - f \\ \boldsymbol{\nabla} u - P_r(\boldsymbol{\beta}) \end{pmatrix}$$



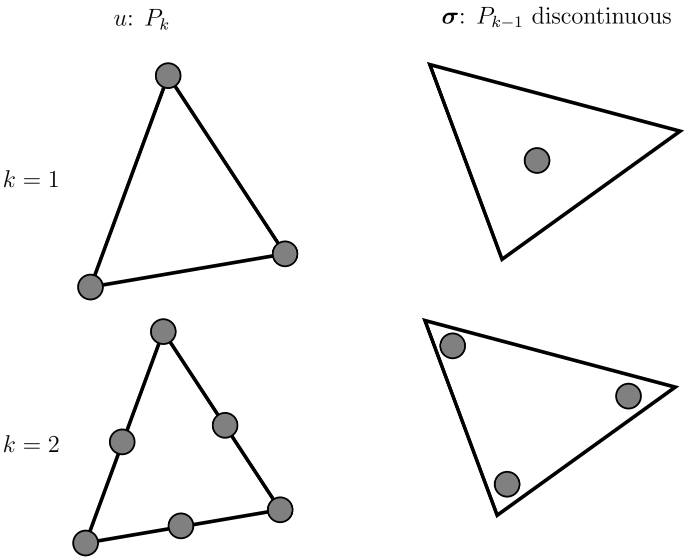
Algorithm

- $k = 0$: χ_0 given
- $k \geqslant 0$: χ_{k-1} known, find $\delta\chi_k$ such that

$$F'(\chi_k).(\delta\chi_k) = -F(\chi_k)$$

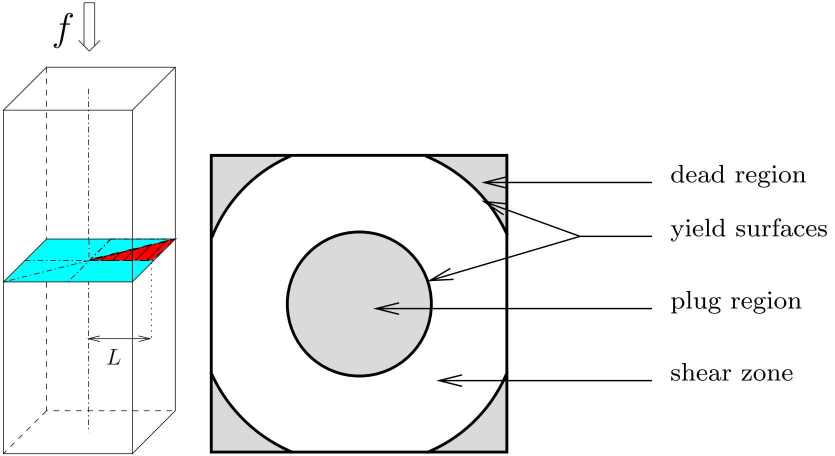
then $$\chi_{k+1} := \chi_k + \delta\chi_k$$

FEM approximation

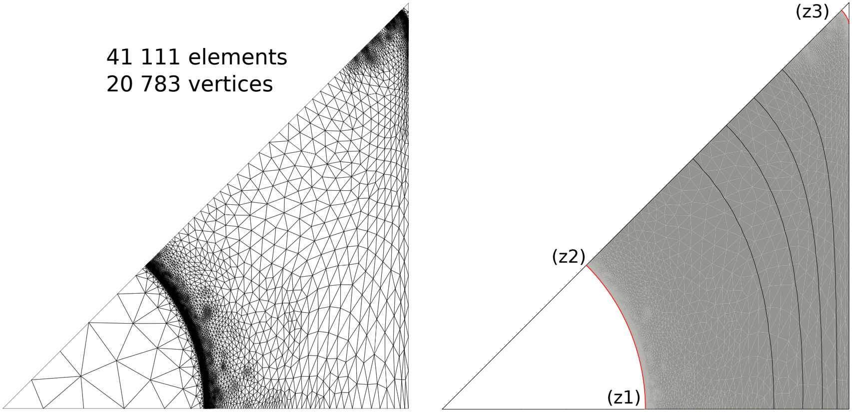


Results: flow in a tube with square section

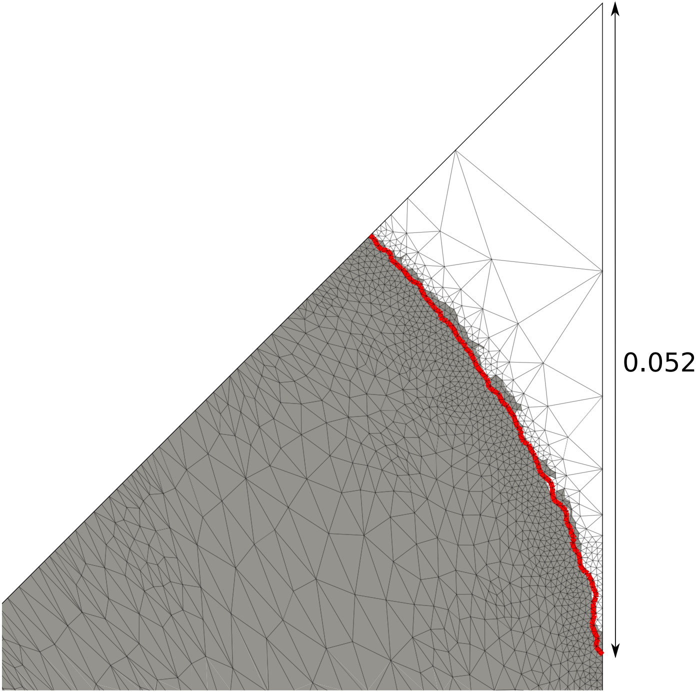
$$Bi = \frac{2\sigma_0}{Lf}$$



$Bi = 0.5$, $n = 0.5$



Comparison with LA method: yield surface in red

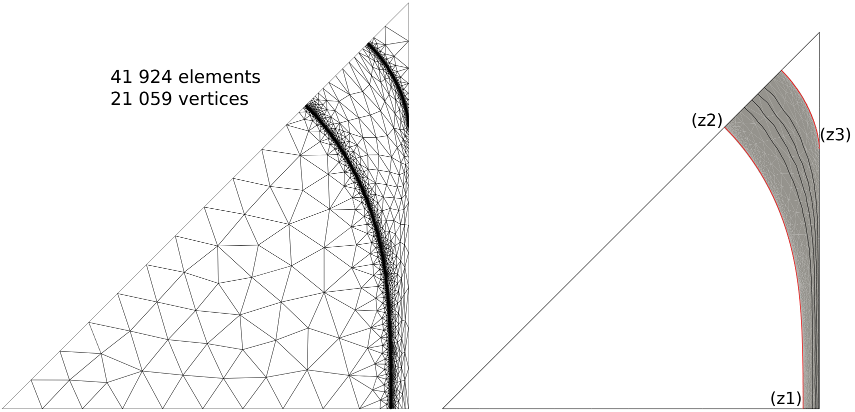

0.052

Approching the arrested state

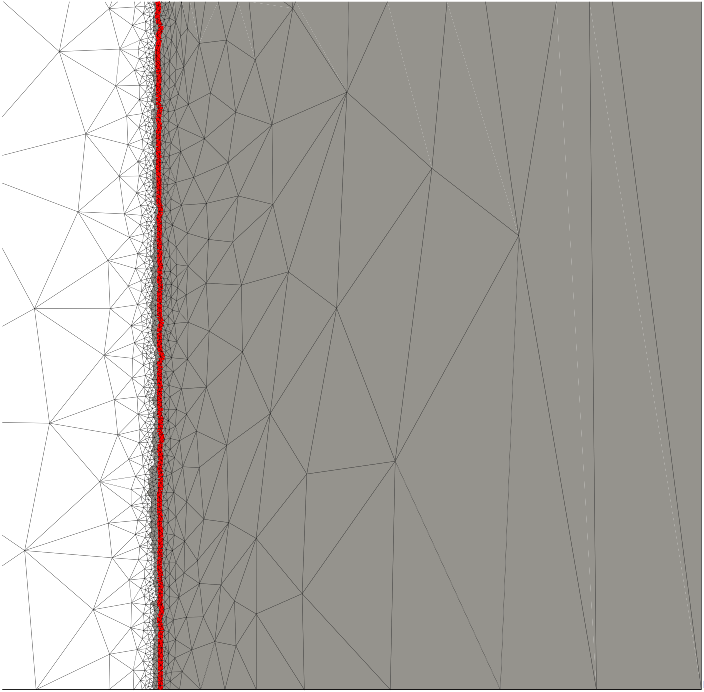
$$Bi = \frac{2\sigma_0}{Lf} \quad \longrightarrow \quad Bi_c = \frac{4}{2+\sqrt{\pi}} \approx 1.0603178\ldots$$

$\Rightarrow$ test with $Bi = 1$

$Bi = 1$, $n = 0.5$



Comparison with LA method: yield surface in red

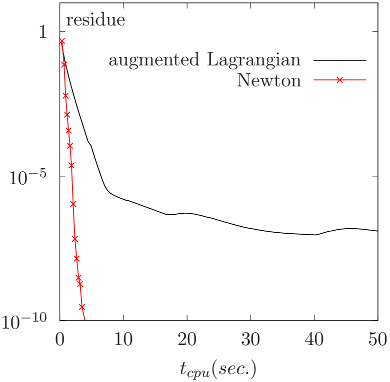


Preliminary conclusion

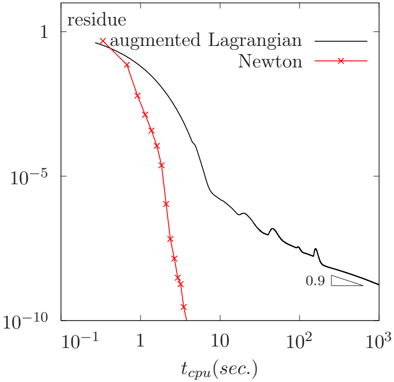
- As accurate as the AL method
- Is it really faster ?

CPU comparison with former method #2

semi-log



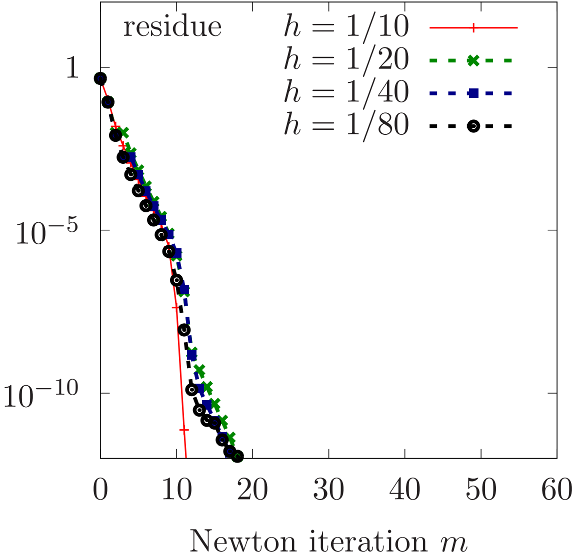
log-log



Speedup

Bi	mesh size	AL method	Newton	Speedup
0.5	41 111	41 hrs	404 sec	350
1	41 924	52 hrs	110 sec	1 700

Mesh-invariant convergence



$Bi = 0.5, \; n = 0.3$

Conclusion

fast-and-accurate

- As **accurate** as the AL method
- Much more **faster**:
  quadratic convergence: $r_{n+1} \approx \alpha\, r_n^2$ insteed of $r_n \approx c\, n^{-1}$

Perspectives

- Apply to more complex flow problems: obstacle, 3D
- Extend to elastoviscoplastic fluids, granular $\mu(I)$

More reading

paper Saramito (2016), A damped Newton algorithm for computing viscoplastic fluid flows, JNNFM

book Saramito (2017). *Complex fluids: modeling and algorithms*, Springer

code Saramito (2017) Rheolef FEM C++ library
Free software (GPL licence)
Source & binaries (Debian, Ubuntu, Mint...)
http://www-ljk.imag.fr/membres/Pierre.Saramito/rheolef