# Consistent equations for open-channel flows

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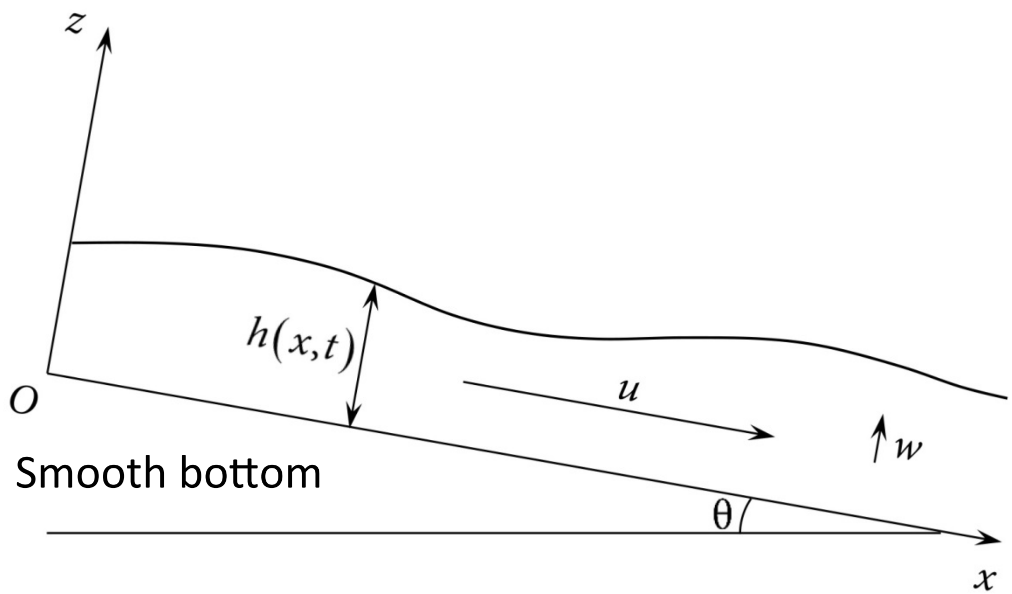
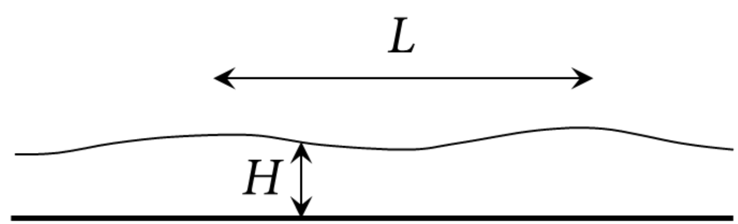
## Turbulent open-channel flows

$10^4 < Re < 10^7$




Shallow water $\varepsilon = \dfrac{H}{L}$ $\varepsilon = 1$



# Energy equation and momentum equation

Mass conservation
Newton's 2nd Law
Work-Energy Theorem (Theorem of the kinetic energy)

} averaged over the depth

→ { Mass equation (M)
Momentum equation (Q)
Energy equation (E)

$$m\frac{\mathrm{d}\vec{\mathrm{v}}}{\mathrm{d}t} = \vec{F}_{ext}$$

(E) and (Q) are independent equations.

$$\frac{\mathrm{d}E_c}{\mathrm{d}t} = P_{ext} + P_{\mathrm{int}}$$

Ext : Weight, Friction on the bottom, air at the free surface
Int : Viscosity or turbulence effects inside the flow

No-slip condition at the bottom ⟶ The wall friction does not do work !

$\vec{F}_{ext}$ *x*-component of the weight
Friction on the bottom
but not viscosity

$P_{ext}$ Power of the x-component of the weight

$P_{int}$ Dissipation power due to viscosity, turbulence
But no power from the friction on the bottom

Dissipation power $\overset{?}{=}$ Friction x average velocity Not always !

⟶ 3 independent equations : (M), (Q), (E)

Models with three independent variables : ok $h, U, \varphi$

Models with two independent variables : (E) and (Q) must be redundant. $h, U(q)$

compatibility condition :

$$(\mathrm{M}) + (\mathrm{Q}) \Rightarrow (\mathrm{E})$$

$$(\mathrm{M}) + (\mathrm{E}) \Rightarrow (\mathrm{Q})$$

# Mathematical structure

Important for the numerical resolution of the equations

- Hyperbolic equations in conservative form
- Source terms : relaxation
- Mass, momentum, energy
- Entropy

Structure of the Euler equations of compressible fluids

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}$$

$$\mathbf{U} = \begin{bmatrix} h \\ hU \\ he \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} hU \\ hU^2 + P \\ hU\left(e + P/h\right) \end{bmatrix}$$

$\mathbf{S}$ : relaxation

# Turbulent open-channel flows

Turbulence Model

Turbulent viscosity          Mixing-length model

Prandtl $L_m = \kappa z$          Van Driest correction $L_m = \kappa z\left(1 - e^{-\frac{z^+}{A^+}}\right)$

$$z^+ = \frac{z u_b}{\nu}\,; \quad u_b = \sqrt{\frac{\tau_b}{\rho}}\,.$$

Two dimensionless systems :

- shallow-water scaling (external layer)
- Viscous scaling (internal layer)

$$\mathrm{Re}_{ML} = \frac{1}{\kappa^2} \qquad \eta = \frac{Re_{ML}}{Re} = O\left(\varepsilon^{2+m}\right), \quad m > 0 \qquad Re = \frac{h_n U_n}{\nu}$$

Matching procedure in a buffer layer or overlap layer $\sqrt{\eta}\, b_1 h < z < \sqrt{\eta}\, b_2 h$

$$b_1 > 0\,; \quad b_2 > 0$$

Constitutive law $\boldsymbol{\tau} = 2\rho(\nu + \nu_T)\mathbf{D}$

$$\nu_T = \sqrt{2}\, L_m^2 \|\mathbf{D}\| \qquad \|D\|^2 = 2\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2$$

## Shallow-water scaling

$$x' = \frac{x}{L}; \quad z' = \frac{z}{h_n}; \quad u' = \frac{u}{u_n}; \quad w' = \frac{w}{\varepsilon u_n}; \quad p' = \frac{p}{\rho g h_n}; \quad t' = \frac{t u_n}{L}. \qquad h' = \frac{h}{h_n}$$

$$L_m' = \frac{L_m}{\kappa h_n} \qquad \mathbf{D}' = \frac{h_n}{u_n}\mathbf{D} \qquad \|\mathbf{D}'\| = \frac{1}{\sqrt{2}}\left|\frac{\partial u'}{\partial z'}\right| + O(\varepsilon^2) \qquad \nu_T = \kappa^2 h_n u_n \sqrt{2} L_m'^2 \|\mathbf{D}'\|$$

$$\nu_e = \kappa^2 h_n u_n$$

$$L_m' = z'\left[1 - \exp\left(-\frac{z'\sqrt{\tau_b'}}{\eta\kappa A^+}\right)\right]; \; z' \qquad \boldsymbol{\tau}' = \frac{\boldsymbol{\tau}}{\rho\kappa^2 u_n^2} \qquad Re_{ML} = \frac{h_n u_n}{\nu_e} = \frac{1}{\kappa^2}$$

$$\nu_{eff}' = \frac{\nu + \nu_T}{\nu_e} = \eta + \sqrt{2}\, L_m'^2 \|\mathbf{D}'\| \qquad \eta = \frac{Re_{ML}}{Re} = O\left(\varepsilon^{2+m}\right), \quad m > 0$$

$$= z'^2\left|\frac{\partial u'}{\partial z'}\right| + O(\varepsilon^2)$$

## Viscous scaling

$$\tilde{x} = x' \; ; \quad \tilde{t} = t' \; ; \quad \tilde{u} = u' \; ; \quad \tilde{p} = p' \; ; \quad \tilde{\boldsymbol{\tau}} = \boldsymbol{\tau}'.$$

Magnification : $\tilde{z} = \dfrac{z'}{\eta} \; ; \quad \tilde{w} = \dfrac{w'}{\eta} \; ; \quad \tilde{h} = \dfrac{h'}{\eta}.$ $\quad \tilde{L}_m = L'_m/\eta. \quad \tilde{\boldsymbol{D}} = \eta \boldsymbol{D}'.$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0$$

$$\frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}} = O(\eta),$$

$$\frac{\partial \tilde{p}}{\partial \tilde{z}} = O(\eta),$$

$$\tilde{L}_m = \tilde{z}\left[1 - \exp\left(-\frac{\tilde{z}\sqrt{\tilde{\tau}_b}}{\kappa A^+}\right)\right]$$

$$\left\|\tilde{\boldsymbol{D}}\right\| = \frac{1}{\sqrt{2}}\left|\frac{\partial \tilde{u}}{\partial \tilde{z}}\right| + O\left(\varepsilon^2 \eta\right)$$

$$\tilde{\nu}_{eff} = 1 + \sqrt{2}\tilde{L}_m^2 \left\|\tilde{\boldsymbol{D}}\right\|$$

# Procedure

$$u' = u'_0 + \varepsilon u'_1 + O\left(\varepsilon^2\right)$$

$$\tau'_{xz} = \tau'_0 + \varepsilon \tau'_1 + O\left(\varepsilon^2\right)$$

Momentum equation in the *Ox* direction

$$\frac{\partial u'}{\partial t'} + \frac{\partial u'^2}{\partial x'} + \frac{\partial u' w'}{\partial z'} = \frac{\kappa^2}{\varepsilon}\left(\lambda + \frac{\partial \tau'_{xz}}{\partial z'}\right) - \frac{1}{F^2}\frac{\partial p'}{\partial x'} + O(\varepsilon) \longrightarrow \tau'_{xz}$$

Constitutive law

$$\boldsymbol{\tau}' = 2\nu'_{eff}\mathbf{D}' \longrightarrow \frac{\partial u'}{\partial z'} \longrightarrow u' \longrightarrow U' = \langle u' \rangle$$

$$u' = U' + u^*$$

$$\varphi' = \frac{\langle u^{*2} \rangle}{h'^2}$$

Enstrophy

Mass

$$\frac{\partial u'}{\partial x'} + \frac{\partial w'}{\partial z'} = 0 \longrightarrow w'$$

Momentum equation in the *Oz* direction

$$\frac{\partial p'}{\partial z'} = -\cos\theta + O(\varepsilon) \longrightarrow p'$$

# Approximation of weakly-sheared flows

Teshukov (2007)

$$u^* = O\left(\varepsilon^{\beta}\right), \quad 0 < \beta < 1$$

$$\varphi = O\left(\varepsilon^{2\beta}\right)$$

$$\left\langle u^{*3} \right\rangle = O\left(\varepsilon^{3\beta}\right)$$

Present approach

$$\mu = -\frac{2}{\ln \eta} \qquad \varepsilon < \mu^2 < \mu < 1 \qquad \eta = \frac{Re_{ML}}{Re} = O\left(\varepsilon^{2+m}\right)$$

$$u^* = O\left(\mu\right)$$

$$\varphi' = O\left(\mu^2\right)$$

$$\left\langle u^{*3} \right\rangle = O\left(\mu^3\right)$$

Asymptotic expansion in both coordinates + Matching procedure in an overlap layer leads to complete profile at order one (in O($\varepsilon$) )

# Equations averaged over the depth

$$\frac{\partial h'}{\partial t'} + \frac{\partial h'U'}{\partial x'} = 0 \qquad \text{(M)}$$

$$\frac{\partial h'U'}{\partial t'} + \frac{\partial}{\partial x'}\left[h'U'^2 + P'\right] = \frac{\kappa^2}{\varepsilon}\left[\lambda h' - \tau_{xz}(0)\right] + O(\varepsilon) \qquad \text{(Q)}$$

$$\frac{\partial h'e'}{\partial t'} + \frac{\partial}{\partial x'}\left[h'U'\left(e' + \frac{P'}{h'}\right)\right] = \frac{\kappa^2}{\varepsilon}\left(\lambda h'U' - W\right) + O\left(\mu^3\right) + O(\varepsilon) \qquad \text{(E)}$$

$$e' = \frac{1}{2}\left(U'^2 + h'^2\varphi' + \frac{h'\cos\theta}{F^2}\right) \qquad P' = h'^3\varphi' + \frac{h'^2\cos\theta}{2F^2}$$

$$W = \tau'_{xz}(0)U + \varepsilon\mu^2\frac{\left(\lambda_0 h'\right)^{3/2}}{3}\frac{\partial h'}{\partial x'}\left(\frac{5}{3} - 4\ln 2\right) + O\left(\varepsilon\mu^3\right)$$

# Saint-Venant Equations

Approximation of $O(1)$

(Q) And (E) are redundant.

$$\frac{\partial h'U'}{\partial t'} + \frac{\partial}{\partial x'}\left[h'U'^2 + \frac{h'^2\cos\theta}{2F^2}\right] = -\kappa^2\tau_1'(0) + O(\varepsilon) + O(\mu^2)$$

$$\frac{\partial h'U'}{\partial t'} + \frac{\partial}{\partial x'}\left[h'U'^2 + P\right] = \frac{\kappa^2}{\varepsilon}\left(\lambda h' - \frac{C_f}{\kappa^2}U'|U'|\right) + O(\varepsilon) + O(\mu^2)$$

$$C_f = \frac{\mu^2\kappa^2}{C^2(\mu)} \qquad C(\mu) = 2 + \mu\left(R + 3\ln 2 - \frac{11}{3} + \ln M\right) \qquad M = 2\mu\sqrt{\lambda_0 h'^3}$$

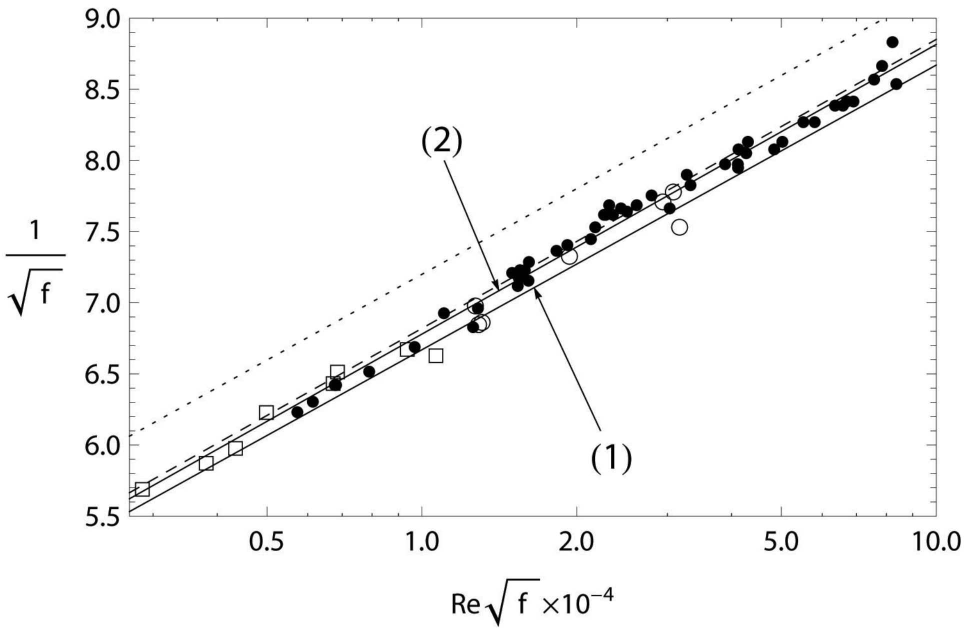
$$R = \int_0^\infty \frac{d\xi}{1+\sqrt{1+\xi^2\left(1-e^{-\xi/A}\right)^2}} - \int_0^\infty \frac{d\xi}{1+\sqrt{1+\xi^2}} \qquad A = 2\kappa A^+$$

$$\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2\cos\theta}{2}\right) = gh\sin\theta - C_f U|U|$$

Darcy friction factor $f = 8C_f$

$$\frac{1}{\sqrt{f}} = \frac{\ln 10}{2\kappa\sqrt{2}} \lg\left(Re_H\sqrt{f}\right) + \frac{1}{2\kappa\sqrt{2}}\left(R + \frac{\ln 2}{2} - \frac{11}{3} + \ln\kappa\right)$$

$Re_H = 4Re$




Tracy & Lester (1961) experiments

Brock (1966, 1967) experiments

(1) $\kappa = 0.41;\quad A^+ = 26$

(2) $\kappa = 0.40;\quad A^+ = 27$

Dotted : Karman-Prandtl law (Colebrook-White relation for smooth pipes)

Dashed : Tracy & Lester law

$R = 2.70$

$$\frac{1}{\sqrt{f}} = 2.035\lg\left(Re_H\sqrt{f}\right) - 1.36$$

$$f = \frac{1.28}{\left(0.887 + \ln\frac{\sqrt{gh^3\sin\theta}}{\nu}\right)^2}$$

# Three-equations model

$$\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0$$

$$\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x}\left[hU^2 + P\right] = \left(1 - \alpha_1\sqrt{C_f}\right)\left(gh\sin\theta - C_f U|U|\right) + \left(\alpha - \alpha_1 - \alpha\alpha_1\sqrt{C_f}\right)\sqrt{C_f}\left(\alpha' h^2\varphi - gh\sin\theta\right)$$

$$\frac{\partial he}{\partial t} + \frac{\partial}{\partial x}\left[hU\left(e + \frac{P}{h}\right)\right] = \left(1 - \alpha\sqrt{C_f}\right)\left(gh\sin\theta - C_f U|U|\right)U - \alpha^2 C_f\left(\alpha' h^2\varphi - gh\sin\theta\right)U$$

$$e = \frac{1}{2}\left(U^2 + h^2\varphi + gh\cos\theta\right) \qquad P = h^3\varphi + \frac{gh^2\cos\theta}{2} \qquad \alpha\ ;\ 7.93\ ; \quad \alpha'\ ;\ 0.217\ ; \quad \alpha_1\ ;\ 1.58\,.$$

$$\frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} = -2\left(\alpha - \alpha_1\right)\frac{\sqrt{C_f}}{h^2}U\left[gh\sin\theta - C_f U|U| + \left(1 + \alpha\sqrt{C_f}\right)\left(\alpha' h^2\varphi - gh\sin\theta\right)\right]$$

Characteristic velocities :

Saint-Venant

$U \pm c$

$c = \sqrt{gh\cos\theta}$

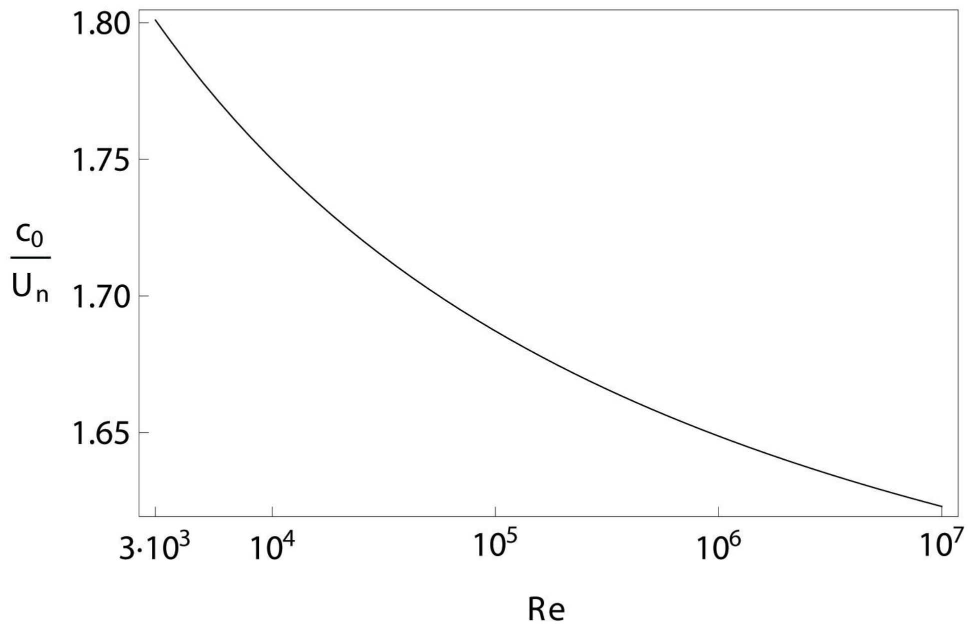
Three-equations model

$U \pm c\ ; \quad U$

$c = \sqrt{gh\cos\theta + 3h^2\varphi}$

# Velocity of the kinematic waves

$$\frac{c_0}{U_n} = \frac{3}{2}\left(1 + \frac{\sqrt{C_n}}{\kappa}\right)$$

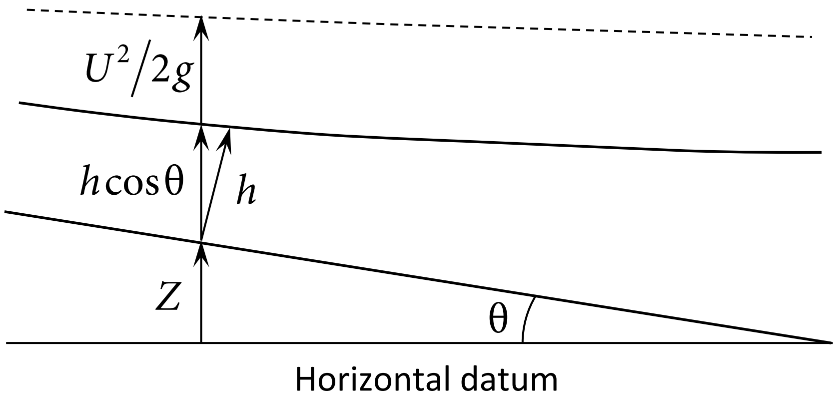



With a constant friction factor $\frac{c_0}{U_n} = \frac{3}{2}$

With the law of Gauckler-Manning-Strickler $\frac{c_0}{U_n} = \frac{5}{3}$

# Total head

Classically : $H = Z + h\cos\theta + \dfrac{U^2}{2g}$




Head loss :

$$H_2 - H_1 = -\int_{x_1}^{x_2} \frac{\tau_{xz}(0)}{\rho g h}\,\mathrm{d}x \qquad \text{or} \qquad H_2 - H_1 = -\int_{x_1}^{x_2} \frac{W}{\rho g h U}\,\mathrm{d}x$$

Friction or energy dissipation ?

Friction slope
Energy slope

Shape factors

Boussinesq $\beta_B = \dfrac{\langle u^2 \rangle}{U^2}$

$$\beta_B \frac{U_2^2}{2g} + h_2\cos\theta + Z_2 = \beta_B \frac{U_1^2}{2g} + h_1\cos\theta + Z_1 - \int_{x_1}^{x_2} \frac{\tau_{xz}(0)}{\rho g h}\,\mathrm{d}x$$

Coriolis $\alpha_C = \dfrac{\langle u^3 \rangle}{U^3}$

$$\alpha_C \frac{U_2^2}{2g} + h_2\cos\theta + Z_2 = \alpha_C \frac{U_1^2}{2g} + h_1\cos\theta + Z_1 - \int_{x_1}^{x_2} \frac{W}{\rho g h U}\,\mathrm{d}x$$

At $O\left(\mu^0\right)$, the friction slope and the energy slope are equal and $\alpha_C = \beta_B = 1$ (Saint-Venant).

At $O\left(\mu^2\right)$ $\quad H = Z + h\cos\theta + \dfrac{U^2}{2g} + \dfrac{3h^2\varphi}{2g}$ $\qquad H_2 - H_1 = -\displaystyle\int_{x_1}^{x_2} \frac{W}{\rho g h U}\,\mathrm{d}x$

Head loss : energy slope only !

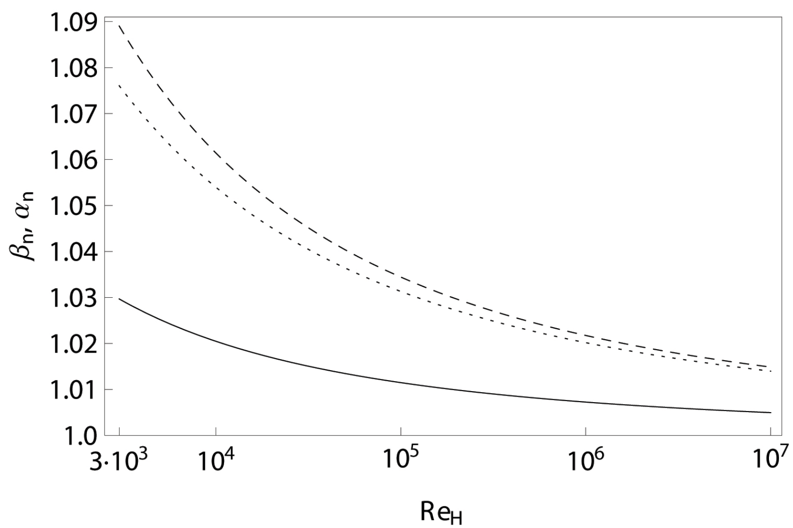
$$\beta_B = 1 + \frac{h^2\varphi}{U^2}$$

$$\alpha_C \;;\; 1 + 3\frac{h^2\varphi}{U^2}$$

Normal values :

$$\beta_B = 1 + \frac{C_n}{\alpha'}$$

$$\alpha_C \;;\; 1 + 3\frac{C_n}{\alpha'}$$



# Backwater curves

Saint-Venant :

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \mathrm{tg}\,\theta \frac{1-\left(h_n/h\right)^3}{1-\left(h_c/h\right)^3}$$

$$K_1 = 1 + \left(2\alpha - 3\alpha_1\right)\sqrt{C_f}$$

Three-equation model :

$$K_2 = \sqrt{C_f}\left[3\left(\alpha - \alpha_1\right) + \alpha\left(2\alpha - 3\alpha_1\right)\sqrt{C_f}\right]$$

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{K_1\left(gh\sin\theta - C_f\, q^2/h^2\right) + K_2\left(\alpha' h^2\varphi - gh\sin\theta\right)}{gh\cos\theta + 3h^2\varphi - q^2/h^2}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = -2\left(\alpha - \alpha_1\right)\frac{\sqrt{C_f}}{h^3}\left[gh\sin\theta - C_f\frac{q^2}{h^2} + \left(1 + \alpha\sqrt{C_f}\right)\left(\alpha' h^2\varphi - gh\sin\theta\right)\right]$$

Boundary conditions

Supercritical case : $h$ and $\varphi$ upstream

Subcritical case : $h$ downstream and $\varphi$ upstream

Shooting method

# Backwater curves

$$X = \frac{x}{h_n}; \quad h' = \frac{h}{h_n}; \quad \varphi' = \frac{\varphi h_n}{Fr_n^2 g \cos\theta}; \quad Fr_n^2 = \frac{q^2}{g h_n^3 \cos\theta}$$

Approximation : $\varphi' ; \ \varphi'^* = \dfrac{\sqrt{C_f}}{\alpha'\left(1+\alpha\sqrt{C_f}\right)}\left(\dfrac{\alpha \operatorname{tg}\theta}{Fr_n^2 h'} + \dfrac{\sqrt{C_f}}{h'^4}\right)$

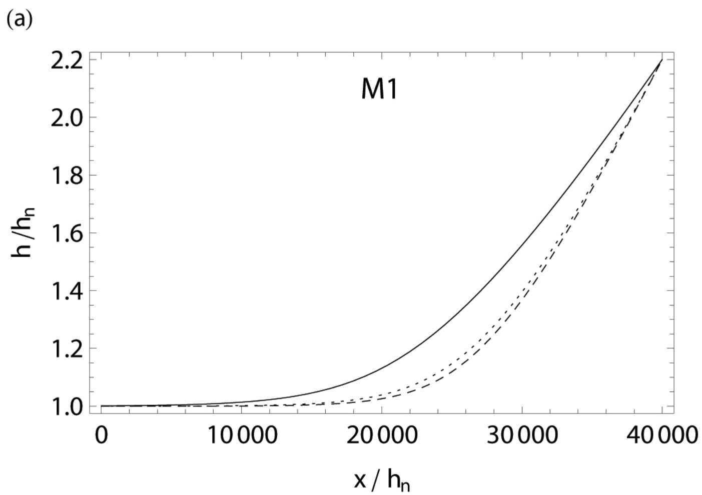
then $\dfrac{dh'}{dX} = \operatorname{tg}\theta \dfrac{1 - C_f/\left(C_n h'^3\right)}{1-\left(h'_c/h'\right)^3} \dfrac{1}{1+\alpha\sqrt{C_f}\left(1+3\operatorname{tg}\theta/\alpha'\right)}$

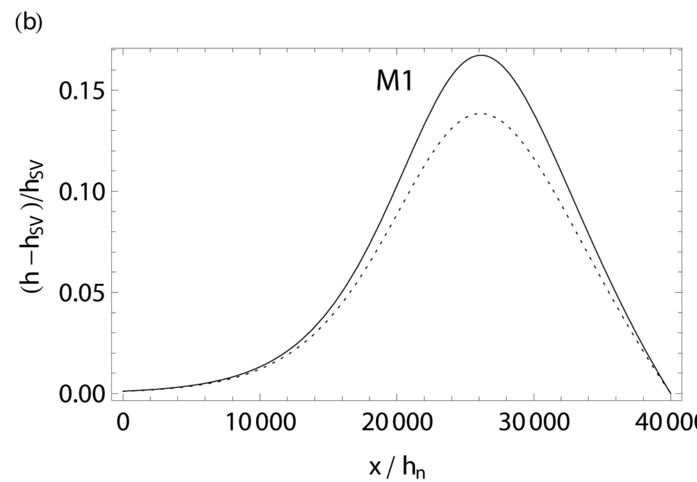
Saint-Venant :

$$\frac{dh'}{dX} = \operatorname{tg}\theta \frac{1-1/h'^3}{1-\left(h'_c/h'\right)^3}$$

Boundary conditions : upstream if supercritical
downstream if subcritical

$Re_H = 10^5$
$C_n = 0{,}00249$
$Fr_n = 0{,}2$
$\operatorname{tg}\theta = 10^{-4}$



# Numerical resolution

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}$$

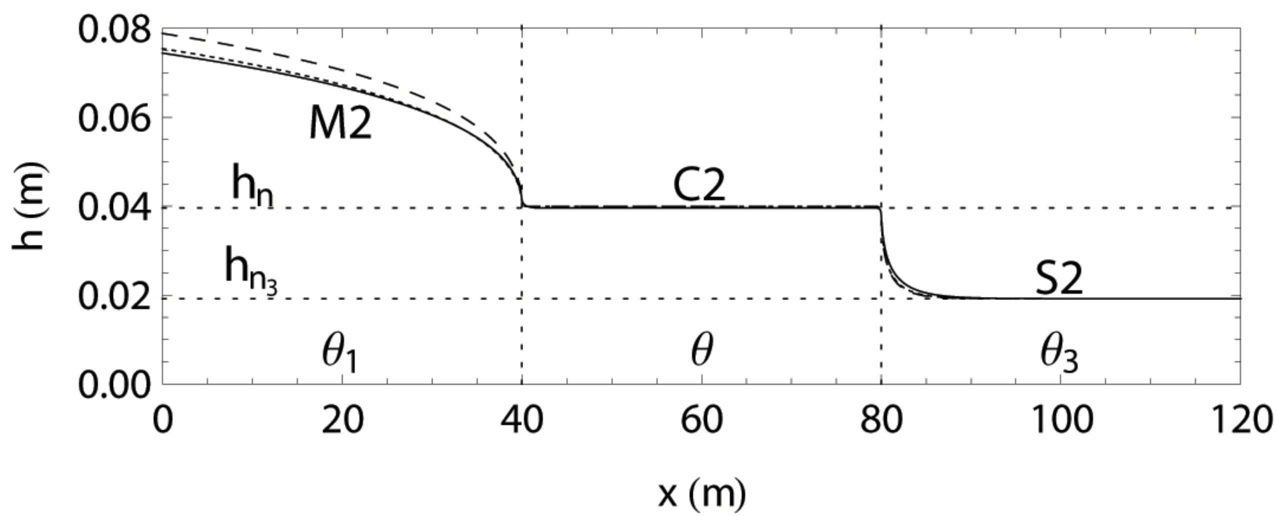
$$\mathbf{U} = \begin{bmatrix} h \\ hU \\ he \end{bmatrix}; \quad F = \begin{bmatrix} hU \\ hU^2 + P \\ hU\left(e + P/h\right) \end{bmatrix}$$

Discretization :

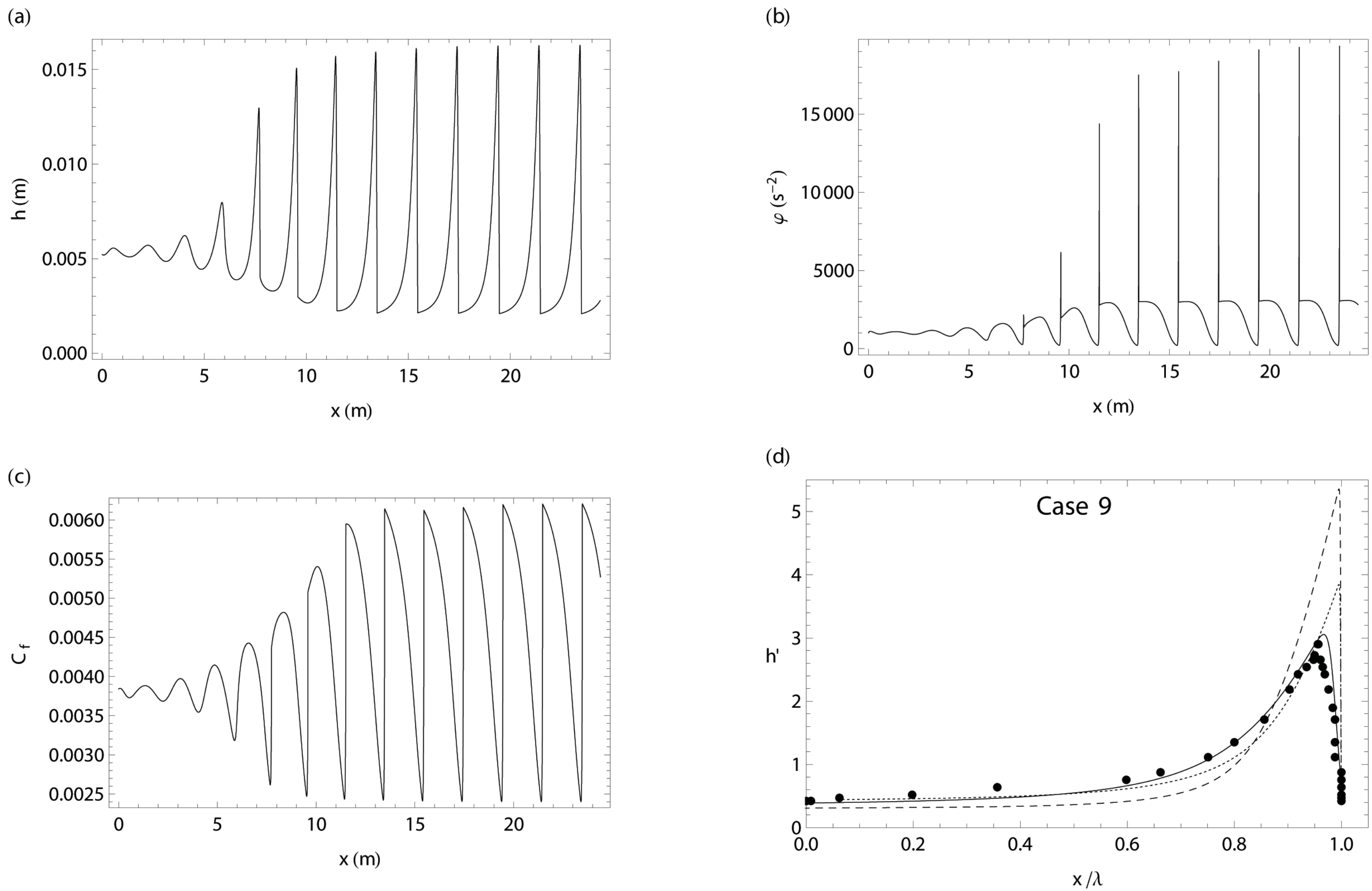
$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i-1/2}^n - \mathbf{F}_{i+1/2}^n\right) + \mathbf{S}_i^n \Delta t$$

Godunov finite-volume method with Riemann solver (Rusanov or HLLC).
Second-order scheme : MUSCL + Heun.

(a)



# Roll waves




Error on wave velocity : 1,1% (SV : 5,7% or 8,3%)
Error on wave amplitude : 7,4% (SV : 37% or 104%)
Error on shock thickness : 24% (SV : 100%)

# Conclusion

Structure of the Euler equations of compressible fluids + source terms

- Model of turbulence
- Approximation of weakly-sheared flows
- No adjustable parameter designed for particular phenomena

The Saint-Venant equations with a uniform velocity and friction are a consistent model.

The energy slope is not equal to the friction slope with a non-flat profile.

The friction is calculated by a local law. For Saint-Venant : constant friction factor.

Next :

- Rough bottom
- Variable bottom
- 2D model
- erosion