Numerical simulation of a depth-averaged Euler system and comparison with analytical solutions

ANGE TEAM : INRIA - UPMC - CNRS - CEREMA

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Contents

The inviscid shallow water model

Incompressible Euler System:

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= G \\ \frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{u}(E + p)) &= 0 \end{aligned}$$

$$E = \frac{u^2 + w^2}{2} + gz$$

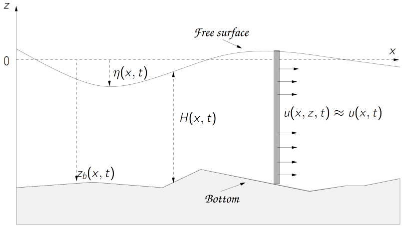
Kinematic boundary conditions

$$\frac{\partial \eta}{\partial t} + u_s \frac{\partial \eta}{\partial x} - w_s = 0$$

$$u_b \frac{\partial z_b}{\partial x} - w_b = 0$$

Boundary pressure $p^a = 0$

$$H = \eta - z_b$$



$$\overline{u}(x,t) = \frac{1}{H}\int_{z_b}^{\eta} u(x,z,t)dz$$

The inviscid shallow water model

Incompressible hydrostatic Euler

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} &= 0 \\
\underbrace{\frac{\partial w}{\partial t} + \frac{\partial wu}{\partial x} + \frac{\partial w^2}{\partial z}}_{\text{non-hydrostatic terms}} + \frac{\partial p}{\partial z} &= -g
\end{aligned}
$$

h,L: characteristic depth and length

Under the shallow water Assumption : $\epsilon = \frac{h}{L} \ll 1$

Approximation by the average

$$u(x,z,t) \approx \overline{u}(x,t)$$

Hydrostatic Pressure

$$\frac{\partial p}{\partial z} = -g$$

Saint-Venant Equation

$$
\begin{aligned}
\frac{\partial H}{\partial t} + \frac{\partial H\overline{u}}{\partial x} &= 0 \\
\frac{\partial H\overline{u}}{\partial t} + \frac{\partial}{\partial x}\left(H\overline{u}^2 + g\frac{H^2}{2}\right) &= -gH\frac{\partial z_b}{\partial x}
\end{aligned}
$$

The Saint-Venant system

Many applications

- Fluvial flow
- Tsunami
- Dam break ..

Figure : Wave propagation

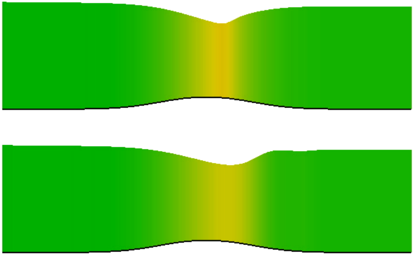
Figure : A dam break

Figure : A Tsunami

Figure : A mascaret

Figure : An hydraulic jump

Problematic : Need more complex model to better simulate phenomena with significant vertical acceleration and dispersion effects.



A depth-averaged Euler system

Non-hydrostatic terms no more neglected
Asymptotic approximation no more required

Euler system $\Rightarrow$

$\varphi = \mathbb{1}_{z_b \leq z \leq \eta}$

$$\left\{ \begin{array}{rcl} \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi u}{\partial x} + \frac{\partial \varphi w}{\partial z} & = & 0 \\ \frac{\partial \varphi u}{\partial t} + \frac{\partial \varphi u^2}{\partial x} + \frac{\partial \varphi uw}{\partial z} + \varphi \frac{\partial p}{\partial x} & = & 0 \\ \frac{\partial \varphi w}{\partial t} + \frac{\partial \varphi uw}{\partial x} + \frac{\partial \varphi w^2}{\partial z} + \varphi \frac{\partial p}{\partial z} & = & -\varphi g \end{array} \right.$$

Depth Averaging $\Rightarrow$

$$\langle f \rangle = \int_{z_b}^{\eta} f \, dz$$

$$p = g(\eta - z) + p_{nh}$$

$$\begin{array}{rcl} \frac{\partial}{\partial t}\langle \varphi \rangle + \frac{\partial}{\partial x}\langle \varphi u \rangle & = & 0 \\ \frac{\partial}{\partial t}\langle \varphi u \rangle + \frac{\partial}{\partial x}\left(\langle \varphi u^2 \rangle + \langle \varphi p \rangle \right) & = & 0 \\ \frac{\partial}{\partial t}\langle \varphi w \rangle + \frac{\partial}{\partial x}\langle \varphi uw \rangle & = & p_{nh}|_b \\ \frac{\partial}{\partial t}\langle \varphi z \rangle + \frac{\partial}{\partial x}\langle \varphi zu \rangle & = & \langle \varphi w \rangle \end{array}$$

Closure relations needed

Closure relations

⇒ Energy balance:

$$\frac{\partial}{\partial t}\langle \varphi E\rangle + \frac{\partial}{\partial x}\langle \varphi u(E+p)\rangle = 0$$

⇒ Minimization of the Energy [Levermore(1997)]

$$\langle \varphi E\rangle = \frac{\langle \varphi u^2\rangle + \langle \varphi w^2\rangle}{2} + \langle \varphi gz\rangle \geq \frac{\langle \varphi u\rangle^2 + \langle \varphi w\rangle^2}{2} + \langle \varphi gz\rangle$$

$$\varphi u = \frac{\langle \varphi u\rangle}{\langle \varphi\rangle} \quad \varphi w = \frac{\langle \varphi w\rangle}{\langle \varphi\rangle}$$

⇒ Closure relations:

$$\langle \varphi u^2\rangle = \frac{\langle \varphi u\rangle^2}{\langle \varphi\rangle} \quad \langle \varphi uw\rangle = \frac{\langle \varphi u\rangle\langle \varphi w\rangle}{\langle \varphi\rangle} \quad \langle \varphi zu\rangle = \langle \varphi z\rangle\frac{\langle \varphi u\rangle}{\langle \varphi\rangle}$$

and,

$$p_{nh}|_b = 2\overline{p}_{nh}$$

REFERENCE:
An energy-consistent depth-averaged Euler system: derivation and properties. ***M-O. Bristeau et al.***
Vertically averaged models for the free surface Euler system. Derivation and kinetic interpretation. ***J. Sainte-Marie***

The depth-averaged model

$$\begin{aligned}
\frac{\partial H}{\partial t} + \frac{\partial H\overline{u}}{\partial x} &= 0 \\
\frac{\partial H\overline{u}}{\partial t} + \frac{\partial}{\partial x}\left(H\overline{u}^2 + g\frac{H^2}{2} + H\overline{p}_{nh}\right) &= -(gH + 2\overline{p}_{nh})\frac{\partial z_b}{\partial x} \\
\frac{\partial H\overline{w}}{\partial t} + \frac{\partial H\overline{wu}}{\partial x} &= 2\overline{p}_{nh} \\
-\frac{H}{2}\frac{\partial H\overline{u}}{\partial x} + \frac{H\overline{u}}{2}\frac{\partial (H + 2z_b)}{\partial x} &= H\overline{w}
\end{aligned}$$

$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial}{\partial x}\left(\overline{u}\left(\overline{E} + g\frac{H^2}{2} + H\overline{p}_{nh}\right)\right) = 0$$

- Definition of the total pressure : $\overline{p} = \overline{p}_h + \overline{p}_{nh}$, $\bar{p}_h = g\frac{H}{2}$
- Saint Venant system + non-hydrostatic terms
- An averaged equation of the momentum equation in z gives the equation in $H\overline{w}$

Comparison with other dispersive models

- Green-Naghdi with $z_b = Cste$

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\overline{u}) = 0$$

$$\frac{\partial (H\overline{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\overline{u}^2 + \frac{g}{2}H^2 + H\overline{p}_{gn}\right) = 0$$

$$\frac{\partial}{\partial t}(H\overline{w}) + \frac{\partial}{\partial x}(H\overline{uw}) = \frac{3}{2}\overline{p}_{gn}$$

$$\bar{w} = -\frac{H}{2}\frac{\partial \bar{u}}{\partial x}$$

- Energy balance

$$\frac{\partial \overline{E}_{gn}}{\partial t} + \frac{\partial}{\partial x}\overline{u}\left(\overline{E}_{gn} + H\overline{p}_{gn}\right) = 0$$

with $\overline{E}_{gn} = \frac{H}{2}\left(\overline{u}^2 + \frac{2}{3}\overline{w}^2\right) + \frac{g}{2}H^2$

See Dena Kazerani's Poster

A Prediction-correction scheme type

(Chorin-Temam)

$$\frac{\partial}{\partial t}\underbrace{\begin{pmatrix} H \\ H\overline{u} \\ H\overline{w} \end{pmatrix}}_{U} + \frac{\partial}{\partial x}\underbrace{\begin{pmatrix} H\overline{u} \\ H\overline{u}^2 + g\frac{H^2}{2} \\ H\overline{wu} \end{pmatrix}}_{F(U)} + \underbrace{\begin{pmatrix} 0 \\ \frac{\partial}{\partial x}(H\overline{p}_{nh}) + \overline{p}_{nh}\frac{\partial z_b}{\partial x} \\ -2\overline{p}_{nh} \end{pmatrix}}_{P}$$

$$= \underbrace{\begin{pmatrix} 0 \\ -gH\frac{\partial z_b}{\partial x} \\ 0 \end{pmatrix}}_{S_b}$$

completed with the incompressibility constraint $div_{sw}(U) = 0$
We denote the flux F:

$$F = \begin{pmatrix} F_H \\ F_{H\overline{u}} \\ F_{H\overline{w}} \end{pmatrix}$$

A Prediction-correction scheme

$$\frac{U^{n+1/2} - U^n}{\Delta t} + \frac{\partial}{\partial x} F(U^n) = S_b^n \qquad (1)$$

$$\frac{U^{n+1} - U^{n+1/2}}{\Delta t} + P^{n+1} = 0 \qquad (2)$$

under the constraint written at the time t^{n+1}:

$$div_{SW}(U^{n+1}) \doteq -H\overline{w}^{n+1} - \frac{H^{n+1}}{2} \frac{\partial (H\overline{u})^{n+1}}{\partial x} + \frac{(H\overline{u})^{n+1}}{2} \frac{\partial (H^{n+1} + 2z_b)}{\partial x} = 0 \qquad (3)$$

Elliptic equation

Equation (3) with (2) gives:

$$-4H^2 \frac{\partial^2 \overline{q}_{nh}}{\partial x^2} + \Lambda \overline{q}_{nh} = \frac{8\sqrt{H}}{\Delta t} div_{SW}(U^{n+1/2})$$

$$\overline{q}_{nh} = \sqrt{H}\overline{p}_{nh} \ , \qquad \Lambda = \Lambda\left(\frac{\partial^j H}{\partial x^j}, \frac{\partial^j z_b}{\partial x^j}\right) , j = 1, 2$$

Prediction step

$$\frac{U^{n+1/2} - U^n}{\Delta t} + \frac{\partial}{\partial x} F(U^n) = S_b^n$$

$\Rightarrow$ Numerical scheme for the Saint Venant system with topography term

$$\begin{aligned} H_i^{n+1/2} &= H_i^n - \sigma(F^n_{H,i+1/2} - F^n_{H,i-1/2}) \\ (H\overline{u})_i^{n+1/2} &= (H\overline{u})_i^n - \sigma(F^n_{H\overline{u},i+1/2} - F^n_{H\overline{u},i-1/2}) \end{aligned}$$

Kinetic Interpretation
+
Hydrostatic reconstruction

$\Rightarrow$ Numerical scheme for the transport equation of $H\overline{w}$

$$(H\overline{w})_i^{n+1/2} = (H\overline{w})_i^n - \sigma(F^n_{H\overline{w},i+1/2} - F^n_{H\overline{w},i-1/2})$$

Correction step

- After computing ${\overline{p}_{nh}}^{n+1}$

$$
\begin{aligned}
H^{n+1} &= H^{n+1/2} \\
(H\overline{u})^{n+1} &= (H\overline{u})^{n+1/2} - \Delta t\left(\frac{\partial}{\partial x}(H\overline{p}_{nh})^{n+1} + 2\overline{p}_{nh}^{n+1}\frac{\partial z_b}{\partial x}\right) \\
(H\overline{w})^{n+1} &= (H\overline{w})^{n+1/2} + 2\Delta t {\overline{p}_{nh}}^{n+1}
\end{aligned}
$$

- Discretisation of the source term and of $H\overline{p}_{nh}$ by a centered finite difference scheme

Properties of the scheme

- **Positivity of H**

CFL condition of the shallow water model $\Rightarrow$ Positivity of $H^{n+1/2}$
Splitting scheme $\Rightarrow H^{n+1} = H^{n+1/2}$

- **The lake at rest**

Lake at rest $\Leftrightarrow H\overline{u} = 0$ and $H + z_b = Cste$
Elliptic equation $\Rightarrow \overline{p}_{nh} = 0$
Verified by the hydrostatic part with the hydrostatic reconstruction.

$\Rightarrow$ We conserve properties of the Saint-Venant system

The elliptic equation

Elliptic equation

$$-4H^2\frac{\partial^2\overline{q}_{nh}}{\partial x^2}+\Lambda\overline{q}_{nh}=\frac{8\sqrt{H}}{\Delta t}div_{SW}(U)$$

$\Rightarrow$ Difficulty to analyse the sign of Λ

- Flat bottom

$$\Lambda=16-2H\frac{\partial^2 H}{\partial x^2}+3(\frac{\partial H}{\partial x})^2$$

- Variable bottom

$$\Lambda=16\left(1+(\frac{\partial z_b}{\partial x})^2\right)-8H\frac{\partial^2 z_b}{\partial x^2}+16\frac{\partial H}{\partial x}\frac{\partial z_b}{\partial x}-2H\frac{\partial^2 H}{\partial x^2}+3(\frac{\partial H}{\partial x})^2$$

Positivity of the total pressure, Behaviour when $H \to 0$

Hydrostatic model : Positivity of the pressure.

$$\overline{p} = \overline{p}_h = g\frac{H}{2}$$

Non-hydrostatic model : How to ensure the positivity?

$$\overline{p} = \overline{p}_h + \overline{p}_{nh} = g\frac{H}{2} + \overline{p}_{nh}$$

⇒ Solution 1

In the elliptic solver, not allow the scheme to degenerate:

$$H_\epsilon = max(H, H_\epsilon)$$

$$-4{H_\epsilon}^2 \frac{\partial^2 \overline{q}_{nh}}{\partial x^2} + \Lambda \overline{q}_{nh} = \frac{8\sqrt{H}}{\Delta t} div_{SW}(U)$$

⇒ Solution 2

Transition to hydrostatic model

$$\begin{aligned}
\frac{\partial \tilde{\rho} H\overline{w}}{\partial t} + \frac{\partial \tilde{\rho} H\overline{wu}}{\partial x} &= 2\overline{p}_{nh} \\
\frac{\partial \tilde{\rho} H}{\partial t} + \frac{\partial \tilde{\rho} H\overline{u}}{\partial x} &= \frac{\tilde{\rho}_M H - \tilde{\rho} H}{\tau} \\
\tilde{\rho}_M &= \begin{cases} 1 \text{ if } p > p_\epsilon \\ 0 \text{ else} \end{cases}
\end{aligned}$$

Propagation of a solitary wave

$$H = H_0 + asech(\frac{x - c_0 t}{l})^2$$

Propagation of a solitary wave

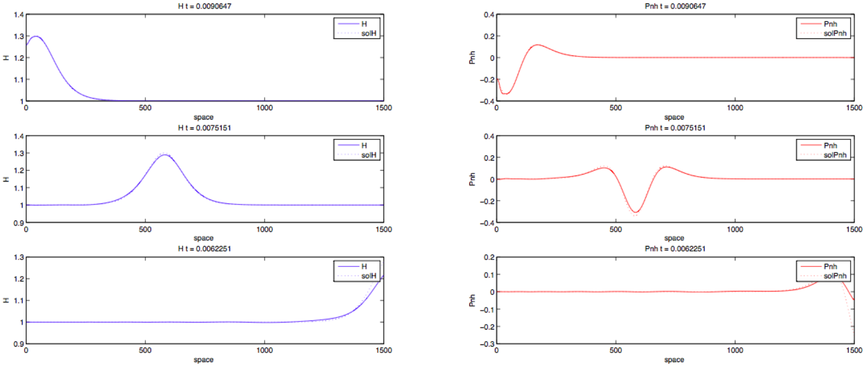


Figure : H and Pnh over the time

A stationary quasi-analytic solution

Stationary solution : $H\overline{u} = Q_0 = Cte$, $H = H(x)$

$$\begin{aligned}
\overline{p}_{nh} &= \frac{Q_0}{2}\frac{\partial \overline{w}}{\partial x} \\
H\overline{w} &= \frac{Q_0}{2}\frac{\partial}{\partial x}(H + 2z_b) \\
\frac{\partial}{\partial x}\left(\frac{Q_0^2}{H} + H\overline{p}\right) &= -(gH + 2\overline{p}_{nh})\frac{\partial z_b}{\partial x}
\end{aligned}$$

- Choosing $w(x) \Rightarrow$ EDO to solve

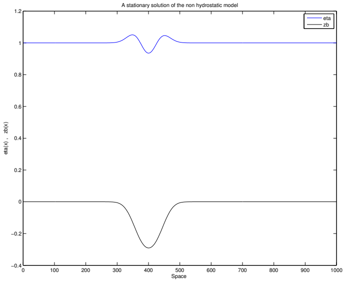
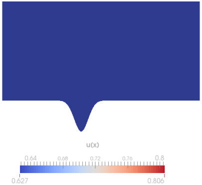


Figure : a particular analytic solution

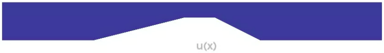
Results - Stationary solution

Initial condition:
$\eta = Cste$
Boundary conditions:
Left : analytical debit q_0
Right : analytical η



Results - Comparison with hydrostatic simulation

Résults - Dispersive effect



Conclusion and outlook

- Properties of the scheme
  - Stability
  - Concistency
  - Convergence
- Validation
  - Analytic solutions
  - Observations
- Error approximation
- Boundary conditions
- 2D,3D model

Thank you !