A well-balanced multiwave relaxation solver for the shallow water magnetohydrodynamic system

Xavier Lhébrard and François Bouchut
xavier.lhebrard@u-pem.fr; francois.bouchut@u-pem.fr

UNIVERSITÉ PARIS-EST MARNE-LA-VALLÉE

SWMHD system

The incompressible MHD system describes the evolution of a charged gas interacting with a magnetic field. In the shallow regime, the SWMHD (Shallow Water MHD) system is relevant.

If the state is described with dependency on only one spatial dimension x and time t, the equations are

$$\begin{aligned}\partial_t h + \partial_x(hu) &= 0,\\ \partial_t(hu) + \partial_x(hu^2 + P) &= -gh\partial_x z,\\ \partial_t(hv) + \partial_x(huv + P_\perp) &= 0,\\ \partial_t(ha) + u\partial_x(ha) &= 0,\\ \partial_t(hb) + \partial_x(hbu - hva) + v\partial_x(ha) &= 0,\end{aligned}$$

with
h: height of the fluid,
$(u,v)^T$: velocity vector
$(a,b)^T$: magnetic field vector

$$P = g\frac{h^2}{2} - ha^2, \quad P_\perp = -hab.$$

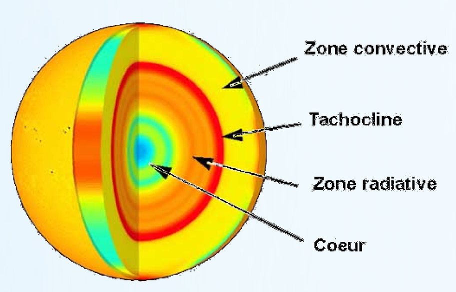
The eigenvalues of the sytem are

$$u - \sqrt{a^2+gh} < u - |a| < u < u + |a| < u + \sqrt{a^2+gh}.$$

Moreover

- u, $u-|a|$, $u+|a|$ are **linearly degenerate**;
- $u \pm \sqrt{a^2+gh}$ are **genuinely non linear**.

Example of application: the solar tachocline
The solar tachocline is a thin layer between radiative and convective zones of the solar interior.



Bibliography

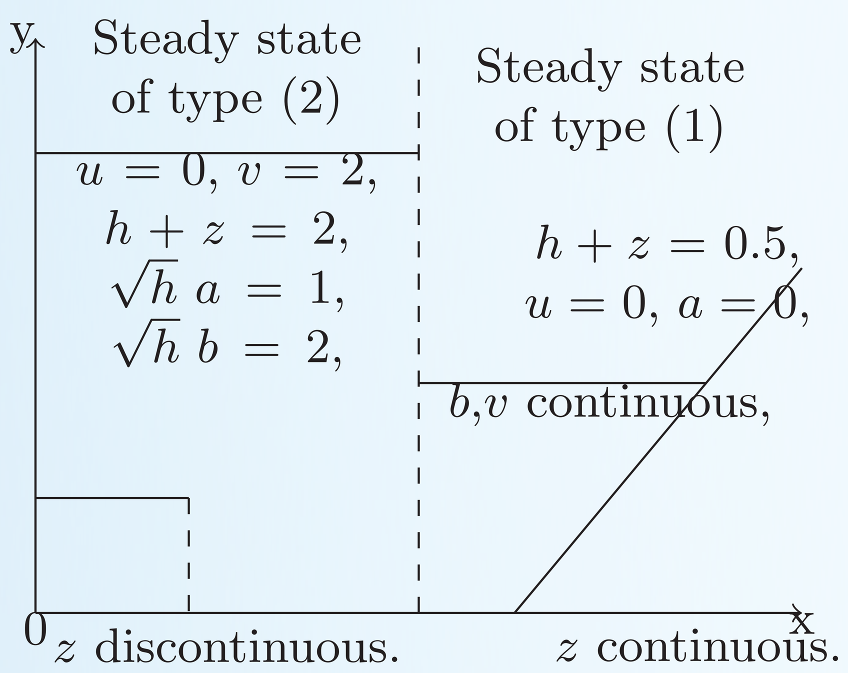
[1] F. Bouchut E. Audusse. A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows. 2004.

[2] K. Waagan F. Bouchut, C. Klingenberg. A multiwave approximate riemann solver for ideal mhd based on relaxation. i: theoretical framework. 2007.

Test case

The space variable x is taken in [0,1]. The test consists of two steady states:

- On [0,1/2), we take initial data corresponding to steady state of type (2).
- On (1/2,1], we take initial data corresponding to steady state of type (1).



Initial data configuration

System with flat bottom

We introduce a **Suliciu type relaxation approximation** for the system with flat bottom. We obtain the following relaxed system:

$$\begin{aligned}\partial_t h + \partial_x(hu) &= 0,\\ \partial_t(hu) + \partial_x(hu^2 + \pi) &= 0,\\ \partial_t(hv) + \partial_x(huv + \pi_\perp) &= 0,\\ \partial_t(ha) + u\partial_x(ha) &= 0,\\ \partial_t(hb) + \partial_x(hbu - hav) + v\partial_x(ha) &= 0,\\ \partial_t(h\pi) + \partial_x(h\pi u) + c^2\partial_x u &= 0,\\ \partial_t(h\pi_\perp) + \partial_x(h\pi_\perp u) + c_a^2\partial_x v &= 0,\\ \partial_t c + u\partial_x c &= 0,\\ \partial_t c_a + u\partial_x(c_a) &= 0.\end{aligned}$$

where π, $\pi_\perp$ are new variables, the relaxed pressures, and c_a, c intended to parametrize the speeds. The eigenvalues of the relaxed sytem are

$$u - \frac{c}{h} < u - \frac{c_a}{h} < u < u + \frac{c_a}{h} < u + \frac{c}{h},$$

and they are all **linearly degenerate**.

Properties of the solver

Using this approach, we are able to prove that:

- under some subcharacteristic conditon and a particular choice of $c_{a,l}$, $c_{a,r}$, c_l, c_r the solver **satisfies a discrete entropy inequality, and preserves positivity of height**;
- it resolves exactly all **contact discontinuities**;
- data with bounded propagation speeds give **finite numerical propagation speed**;
- the **numerical viscosity is sharp**,in the sense that the propagation speeds of the approximate Riemann solver tend to the exact propagation speeds when the left and right states tend to a common value.

Topography treatment

The SWMHD system with non-flat bottom has four linearly degenerate eigenvalues

$$u, \quad u - |a|, \quad u + |a|, \quad 0,$$

respectively called material, left Alfven, right Aflven and topography waves, that can be resonant. We want our scheme to preserve some families of contact discontinuities associated to the 0-wave. Thus we deal with different cases:

- material contact and Alfven contact resonance case. $(u = a = 0)$ satisfying
$$h + z = cst, \quad u = 0, \quad a = 0, \qquad (1)$$
- material contact resonance case ($u = 0$ and $a \neq 0$) satisfying
$$\begin{aligned} u = 0, \quad v = cst, \quad h + z = cst,\\ \sqrt{h}\ a = cst, \quad \sqrt{h}\ b = cst. \qquad (2)\end{aligned}$$

We use the **hydrostatic reconstruction method** and define reconstructed heights

$$h_l^\# = (h_l - (\Delta z)_+)_+, \quad h_r^\# = (h_r - (-\Delta z)_+)_+.$$

We also define reconstructed magnetic states

$$a_l^\# = \kappa_l a_l, \quad b_l^\# = \kappa_l b_l,$$

with $\kappa_l = \min\left(\sqrt{\frac{h_l}{h_l^\#}}, \gamma\right)$, $\gamma \geq 1$.

The numerical fluxes involve the **reconstructed states and suitable correction terms**.

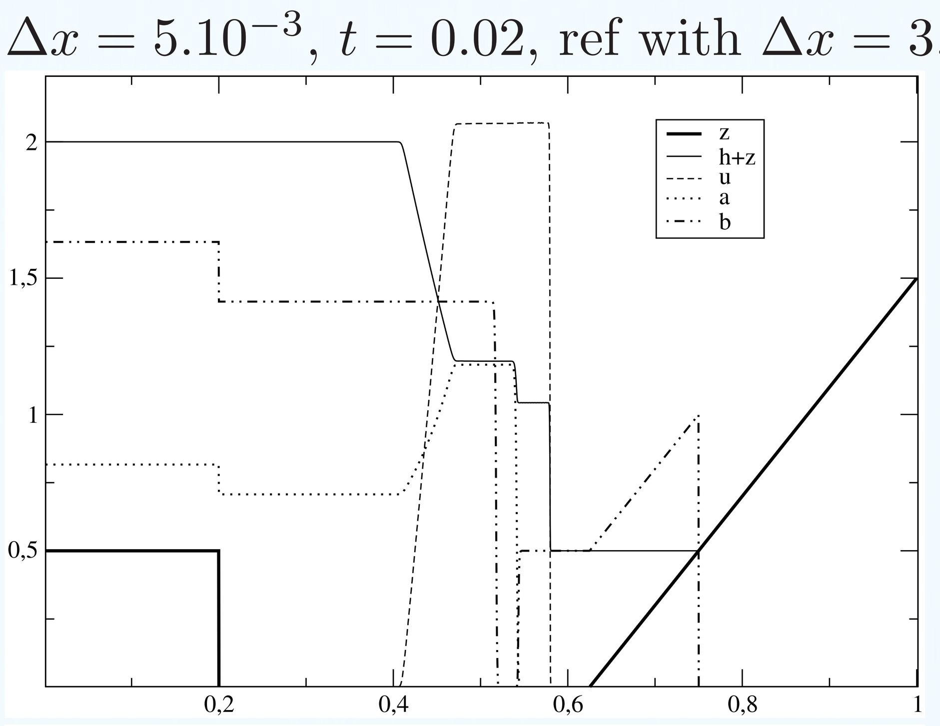
Main results

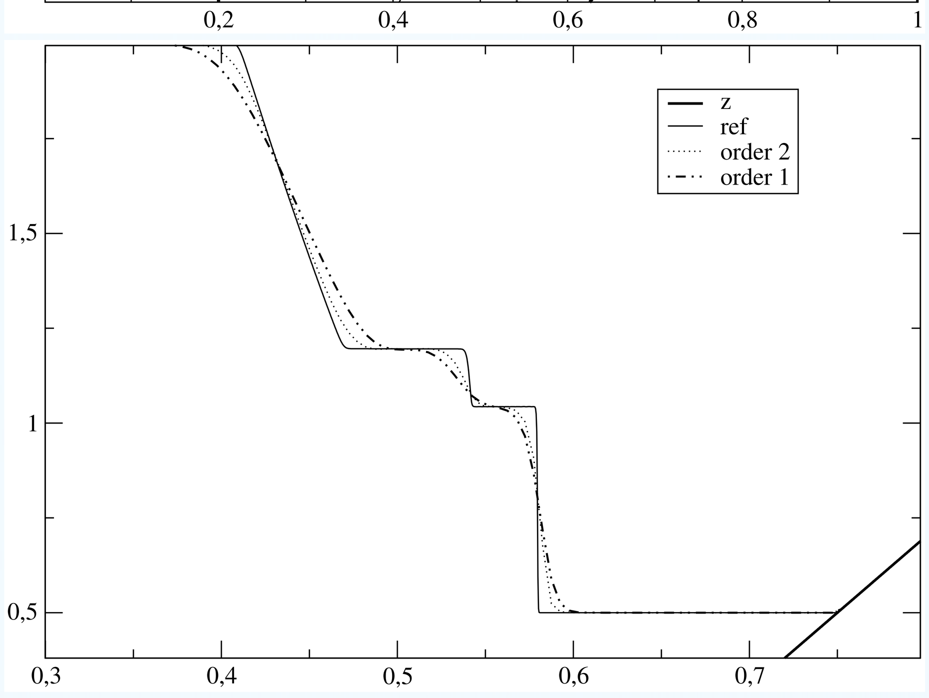
We obtain a scheme that satisfies the following properties:

- it is **consistent**,
- it satisfies a **semi-discrete entropy inequality**,
- it preserves the **nonnegativity of the thickness** of the fluid layer,
- it is **well-balanced**, i.e. it resolves exactly contact discontinuities of type (1) and (2).

Numerical results

$\Delta x = 5.10^{-3}$, $t = 0.02$, ref with $\Delta x = 3.10^{-4}$.





$\Delta x = 3.10^{-4}$, $t = 0.08.$, ref with $\Delta x = 3.10^{-4}$.