

Pierre-Yves Lagrée

"Hydrodynamic and erosion models for flows over erodible beds 1/2"

Chalès 01/07/14

Institut Jean Le Rond ∂'Alembert

to introduce a boundary layer in order to recover the no slip condition 2.3 Averaged Boundary Layers L becaust the lab Hele Shaw. layer separation, one has to do a triple deck theory. The separation of the separation \mathcal{C} D. Doppler T. Loiseleux deal with boundary layer separation is to use the idea of "Interacting exemple in the lab Hele Shaw u

 $s \rightarrow$

 \overline{a}

 \overline{a}

exemple in the lab Hele Shaw D. Doppler T. Loiseleux

- continuum models
- bed load suspension

"sophisticated" models Jackson 00 Charru & Mouilleron Arnoud 02 Ouiremi, Aussillous & Guazzelli 09 Chauchat Medale 10 $q(\theta)$

feel each other collisional grains water bifluids

free fall equilibrium

$$
\frac{4}{3}\pi(\Delta\rho)R^3g = 6\pi\eta RV_s
$$

terminal velocity

$$
V_s = \frac{d^2}{18\eta} (\Delta \rho) g
$$

Stokes velocity

$V_s =$ d^2 18η $(\Delta \rho)g$ $Re =$ d^2 ν ∂u ∂y Reynolds of the particule terminal velocity

Reynolds associated= Galileo

$$
Ga = \frac{(\Delta \rho)gd^3}{\eta^2}
$$

Albert F. Shields (1908–1974) no publication: phD Technischen Hochschule Berlin, 1936

Les lois d'entraˆınement de M. Scipion Gras S tress lerger then a threshold T $>$ T_c Stress larger than a threshold $\tau > \tau_s$

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free fall equilibrium

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$$

terminal velocity

$$
V_s = \frac{d^2}{18\eta} (\Delta \rho) g
$$

characteristic flux

$$
Q_s = \frac{d^3}{\eta} (\Delta \rho) g
$$

Lagrangian dilute A_n

shall discuss in \mathbb{R}^2 (see Sec.15). In particular, the friction is formula (52.15%). In particular, the force on \mathbb{R}^2 Basset historic term t $\sigma_{xy} = -\sqrt{\frac{\eta \rho}{\pi}} \int_{0}^{t} \frac{du(\tau)}{d\tau} \frac{d\tau}{\sqrt{(t-\tau)}};$ $-\infty$

Let us now consider the general case ofan oscillating body with any shape. In the case of added mass/ acceleration

Turbulent Dispersed Multiphase Flow shape. We shall assume that the small assume that the other than the oth terms, so that it may be neglected. The conditions necessary for this procedure to be valid ν below.

S. Balachandar¹ and John K. Eaton² on^2

Turbulent Dispersed Multiphase Flow

S. Balachandar¹ and John K. Eaton²

The continuum-mechanical modeling of sediments depends on its compaction, the particle volume fraction

$$
\phi = \frac{\text{volume of particles}}{\text{volume}}
$$

if ϕ is smaller than the random loose packing with $\phi_m \approx 0.50$ concentrated suspension

if ϕ is larger than the random close packing with $\phi_M \approx 0.65$ the sediment behaves as a poro-elastic solid.

suspension ϕ < ϕ *m* \approx 0.50

suspension dense ϕ < ϕ *m* \approx 0.50 ϕ *m* ≈ 0.50 < ϕ < ϕ *m* ≈ 0.65

grains

suspension dense $φm \approx 0.50 < φ < φ_M \approx 0.65$ ϕ < ϕ *m* \approx 0.50

fixed

grains

grains

suspension ϕ < ϕ *m* \approx 0.50

dense

 $φm ≈ 0.50 < φ < φ_M ≈ 0.65$

fixed

continnum model

Jean le Rond ∂'Alembert 1717-1783

introduction of «continuum media»

Continuum approach

ensemble average distribution function

volumic average

D. Lhuillier R. Jackson

ergodicity…

Continuum approach

ensemble average distribution function

volumic average

decompose : mean+ fluctuation

 $u_i = \langle u_i \rangle + u'_i$ $\langle u_i u_j \rangle = \langle (u_i u_j + u'_i)(\langle u_j u_j + u'_j \rangle) \rangle$ $u_i u_j$ >=< u_i >< u_j > + < $u'_i u'_j$ > +0 < u_i > +0 < u_j > Continuum approach
continuum-mechanical modelling of the fluid of the fluid of this sediment depends of this sediment depends of \mathbf{h} cles surrounded by a liquid and with no interpretation and with no interpretation α

 μ in a solid-liquid description of a solid-liquid mixture, the carrier liquid is μ $\begin{split} \textbf{1} \quad \textbf{1} \quad \textbf{1} \quad \textbf{2} \quad \textbf{2} \quad \textbf{2} \quad \textbf{2} \quad \textbf{3} \quad \textbf{3} \quad \textbf{3} \quad \textbf{4} \quad \textbf{4} \quad \textbf{5} \quad \textbf{5} \quad \textbf{6} \quad \textbf{7} \quad \textbf{7} \quad \textbf{8} \quad \textbf{7} \quad \textbf{8} \quad \textbf{8} \quad \textbf{9} \quad \textbf{1} \quad \textbf{1} \quad \textbf{1} \quad \textbf{1} \quad \textbf{1} \$ $particle\ volume\ fraction\ \phi$

 $\sqrt{2}$ v uluing nagalum of ang generally ngulum v ∙ri $\frac{1}{2}$ \overline{a} \overline{a} $\frac{1}{2}$ the random loose packing with $\frac{1}{2}$ for spin-forms $\frac{1}{2}$ for spin-forms $\frac{1}{2}$ for spin-forms $\frac{1}{2}$ volume fraction of the carrier liquid 1- ϕ

mass densities (ρp for the solid particles and particles and particles and particles and particles and particles and tiow is supposed incompressible flow is supposed incompressible common α is order α in second particles in second particles. flow is supposed incompressible

$$
\bigcirc \bigcirc \bigcirc \qquad \frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{V}^p) = 0,
$$

$$
\nabla \cdot \mathbf{V} = 0, \qquad \mathbf{V} = \phi \mathbf{V}^p + (1 - \phi) \mathbf{V}^f
$$

$$
\text{volume-averaged velocity of the mixture.}
$$

Continuum approach
continuum-mechanical modelling of the fluid of the fluid of this sediment depends of this sediment depends of \mathbf{h} cles surrounded by a liquid and with no interpretation and with no interpretation α

 μ in a solid-liquid description of a solid-liquid mixture, the carrier liquid is μ $\begin{split} \textbf{1} \quad \textbf{1} \quad \textbf{1} \quad \textbf{2} \quad \textbf{2} \quad \textbf{2} \quad \textbf{2} \quad \textbf{3} \quad \textbf{3} \quad \textbf{3} \quad \textbf{4} \quad \textbf{4} \quad \textbf{5} \quad \textbf{5} \quad \textbf{6} \quad \textbf{7} \quad \textbf{7} \quad \textbf{8} \quad \textbf{7} \quad \textbf{8} \quad \textbf{8} \quad \textbf{9} \quad \textbf{1} \quad \textbf{1} \quad \textbf{1} \quad \textbf{1} \quad \textbf{1} \$ $particle\ volume\ fraction\ \phi$

 $\sqrt{2}$ v uluing nagalum of ang generally ngulum v ∙ri $\frac{1}{2}$ θ \overline{a} volume fraction of the carrier liquid 1- ϕ

mass densities (ρp for the solid particles and particles and particles and particles and particles and particles and tiow is supposed incompressible flow is supposed incompressible common α is order α in second particles in second particles. flow is supposed incompressible

 \mathcal{A}

$$
\begin{aligned}\n\bigotimes \mathbf{e} & \frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{V}^p) = 0, \\
\bigotimes \mathbf{V} \cdot \mathbf{V} &= 0 \,, \qquad \mathbf{V} = \phi \mathbf{V}^p + (1 - \phi) \mathbf{V}^f \\
\text{volume averaged velocity of the mixture}\n\end{aligned}
$$

volume-averaged velocity of the mixture.

$$
\qquad \qquad \text{mean value} \qquad \qquad = +
$$

$$
\text{derivative} \qquad \int \quad \phi \rho_p \frac{d_p \mathbf{u}_p}{dt} + \nabla \cdot (\phi \rho_p < \mathbf{u}'_p \otimes \mathbf{u}'_p>)
$$

 \mathcal{A}

 $=$ $\frac{1}{2}$ $\frac{1}{2$

V \overline{d} iv
| total derivative Erosion being an unsteady phenomenon will not be considered hereafter. In

$$
\text{defivative} \qquad \begin{cases} \qquad & \text{if } (1-\phi)\rho_f \frac{d_f \mathbf{u}_f}{dt} + \nabla \cdot ((1-\phi)\rho_f < \mathbf{u}_f' \otimes \mathbf{u}_f') \end{cases}
$$

Continuum approach Johunuum approach
Restierde pheee Dans ces equations les contraintes σpp sont liees aux forces directes Fpp entres les particules (comme les forces colloidales, les forces de collision ou les forces particule phase \Box particule phase

D. Lhuillier

constraints exerted by the fluid constraints exerted by the particules conctrointe euc constraints exerted by the fluid e partic aints exerted by the particules IIes ≫ appropries pour decrire pour de constraints exerte appropries
appropries overted by the portioules

$$
\mathbf{F}^{pf} = n < \oint (\sigma^{f0} + p_f I) \cdot \mathbf{n} ds
$$
\nmean nbr of part./vol.

\n
$$
\mathbf{F}^{pf} = n < \oint (\sigma^{f0} + p_f I) \cdot \mathbf{n} ds
$$
\n
$$
\oint \phi_p \frac{d_p \mathbf{u}_p}{dt} + \nabla \cdot (\phi \rho_p < \mathbf{u}'_p \otimes \mathbf{u}'_p) = \nabla \cdot \sigma^{pp} + \mathbf{F}^{pf} - \phi \nabla p_f + \phi \rho_p \mathbf{g}
$$

de contact) et sa definition generale est σpp = n < fluid phase

les particules (comme les forces colloidales, les forces de collision ou les forces

p
p ⊗ u⁄
p ⊗ u′

^p [⊗] ^u′

 $\frac{1}{\sqrt{1-\frac{1}{2}}}\left(1-\frac{1}{2}\right)$

fluid phase
$$
\overbrace{\nabla \cdot \tau^f \left(\mathbf{F}^{pf} - (1 - \phi) \nabla p_f \right)}^{(1 - \phi) \rho_f g}
$$

−(1 − φ)∇p^f + (1 − φ)ρfg (2)

$$
(1 - \phi)\rho_f \frac{d_f \mathbf{u}_f}{dt} + \nabla \cdot ((1 - \phi)\rho_f < \mathbf{u}'_f \otimes \mathbf{u}'_f) = \nabla \cdot \tau^f - \mathbf{F}^{pf} - (1 - \phi)\nabla p_f + (1 - \phi)\rho_f \mathbf{g}
$$

^p >) = ∇ · σpp + Fpf − φ∇p^f + φρpg (1)

 $p = \frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \right) \right) \right) \, dx}{\int_{0}^{\infty} \frac{1}{2} \left(\int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2$

Continuum approach blackson buoyant decomposition parties the force \mathbf{r} force \mathbf{r} force \mathbf{r} f energy parties inertial exemples \mathbf{r} a la turbulence et ajoute et anno 1980. A la masse a la masse a la masse vista de la masse vista de la masse e

 ach and $bcen$ huovant decomposition Jackson buoyant decomposition

$$
\mathbf{F}^{pf} = \mathbf{F} + \mathbf{F}^v \quad , \quad \tau^f = \tau + \tau^v \ . \qquad \mathbf{F}^v = \phi \nabla \cdot \tau^v + (1-\phi) \mathbf{f}^v
$$

$$
\phi \rho_p \frac{d_p \mathbf{u}_p}{dt} = \phi \rho_f \frac{d_f \mathbf{u}_f}{dt} + \nabla \cdot \sigma^p + \mathbf{r} \cdot \mathbf{v} + \phi (\rho_p - \rho_f) \mathbf{g}
$$
\n
$$
\rho_f \frac{d_f \mathbf{u}_f}{dt} = \nabla \cdot \tau^v - \mathbf{r} \cdot \mathbf{v} + \rho_f \mathbf{g}.
$$

O C interage value of the resultant force exerted by the fluid on the particule Si on pense que l'on est en droit de negliger toutes les quantites liees a la Average value of the resultant force

do into a sum of the buoyancy force and
ately, however, this decomposition is not ontrib
معلم dian te on a particle, namely the buoyancy, is usually regarded as fairly obvious, so f = φρ^f ed
Sverske orage force exerted by the fluid
egarded as fairly obvious, so f is next decomposed without further ado into a sum of the buoyancy force and completely unambiguous. In this connection the description of buoyancy by a Mais si on a des raisons de penser que les fluide du fluide du fluide du fluide du fluide du fluide du fluide
De vites se de vites de vite

The syde [of a ship] being rounde and full, it is the more boyenter a great deale.
(W. Bourne. Treas. for Trav., 1578)

Jackson 00

Continuum approach blackson buoyant decomposition parties the force \mathbf{r} force \mathbf{r} force \mathbf{r} f energy parties inertial exemples \mathbf{r} a la turbulence et ajoute et anno 1980. A la masse a la masse a la masse vista de la masse vista de la masse e

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$$

$$
\phi \rho_p \frac{d_p \mathbf{u}_p}{dt} = \phi \rho_f \frac{d_f \mathbf{u}_f}{dt} + \nabla \cdot \sigma^p + \mathbf{r} \cdot \mathbf{v} + \phi (\rho_p - \rho_f) \mathbf{g}
$$
\n
$$
\rho_f \frac{d_f \mathbf{u}_f}{dt} = \nabla \cdot \tau^v - \mathbf{r} \cdot \mathbf{v} + \rho_f \mathbf{g}.
$$

O C interage value of the resultant force exerted by the fluid on the particule Si on pense que l'on est en droit de negliger toutes les quantites liees a la Average value of the resultant force

> sim $\overline{}$ Z not so simple....

As an example of fallacious equations based on the partition (2.38) we consider the following pair, due to Fessele & Gibilaro (1987): un role important alors il faut retablir occupation il faut retablir occupation il faut retablir occupation i
Description il faut retablir occupation il faut retablir occupation il faut retablir occupation il faut retabl

Jackson 00

Continuum approach est en droit de negliger toutes les quantites de la continuum approach turbulence du fluide et a la masse ajoute du fluide et a la masse ajoute du fluide du fluide du fluide du flui
Une des propriétés ajoute du fluide du

$$
\phi \rho_p \frac{d_p \mathbf{u}_p}{dt} = \phi \rho_f \frac{d_f \mathbf{u}_f}{dt} + \nabla \cdot \sigma^p + \mathbf{f}^v + \phi (\rho_p - \rho_f) \mathbf{g}
$$

$$
\rho_f \frac{d_f \mathbf{u}_f}{dt} = \nabla \cdot \tau^v - \mathbf{f}^v - \nabla p_f + \rho_f \mathbf{g}.
$$

Continuum approach est en droit de negliger toutes les quantites de la continuum approach turbulence du fluide et a la masse ajoute du fluide et a la masse ajoute du fluide du fluide du fluide du flui
Une des propriétés ajoute du fluide du where Eij = 1/2(∂vi/∂xi) is the mean strain rate of the mean strain rate of the mean strain rate of the suspen
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$$
\phi \rho_p \frac{d_p \mathbf{u}_p}{dt} = \phi \rho_f \frac{d_f \mathbf{u}_f}{dt} + \nabla \cdot \sigma^p + \mathbf{f}^v + \phi(\rho_p - \rho_f) \mathbf{g}
$$
\n
$$
\rho_f \frac{d_f \mathbf{u}_f}{dt} = \nabla \cdot \tau^v - \mathbf{f}^v - \nabla p_f + \rho_f \mathbf{g}
$$
\nforce

\n
$$
\bigcirc \bigcirc
$$
\n
$$
\mathbf{f}^v = -\frac{\phi \eta_f}{d^2} \lambda(\phi) (\mathbf{V}^p - \mathbf{V})
$$
\nDarcy

 \mathcal{L} the shear flow. The drag force depends on \mathcal{L}
Continuum approach est en droit de negliger toutes les quantites de la continuum approach turbulence du fluide et a la masse ajoute du fluide et a la masse ajoute du fluide du fluide du fluide du flui
Une des propriétés ajoute du fluide du where Eij = 1/2(∂vi/∂xi) is the mean strain rate of the mean strain rate of the mean strain rate of the suspen
The suspen-to-suspen-to-suspen-to-suspen-to-suspen-to-suspen-to-suspen-to-suspen-to-suspen-to-suspen-to-suspen

$$
\phi \rho_p \frac{d_p \mathbf{u}_p}{dt} = \phi \rho_f \frac{d_f \mathbf{u}_f}{dt} + \nabla \cdot \sigma^p + \mathbf{f}^v + \phi(\rho_p - \rho_f) \mathbf{g}
$$
\n
$$
\rho_f \frac{d_f \mathbf{u}_f}{dt} = \nabla \cdot \tau^v - \mathbf{f}^v - \nabla p_f + \rho_f \mathbf{g}
$$
\nforce

\n
$$
\begin{array}{c}\n\bigcirc \bigcirc \\
\bigcirc \bigcirc \\
\mathbf{f}^v = -\frac{\phi \eta_f}{d^2} \lambda(\phi) (\mathbf{V}^p - \mathbf{V}) - \phi \nabla (\eta_f \nu(\phi) \dot{\gamma})\n\end{array}
$$
\nDarcy

\nResuspension

$$
p(x + \Delta x) - p(x) \simeq -\Delta x \frac{\eta}{R^2} U
$$

Resuspension Leighton Acrivos

free fall equilibrium in shear

$$
-\frac{4}{3}\pi(\Delta\rho)R^3g = (\pi\eta R^2)(\dot{\gamma}(y+R)\phi(y+R) - \dot{\gamma}(y)\phi(y))
$$

Ofree fall flux due to shear
terminal velocity

$$
V_s = \frac{d^2}{18\eta}(\Delta\rho)g
$$

$$
\frac{4}{3\eta}(\Delta\rho)R^g + (R^2)\dot{\gamma}(y)\frac{\partial}{\partial y}\phi = 0
$$

Continuum approach est en droit de negliger toutes les quantites de la continuum approach turbulence du fluide et a la masse ajoute du fluide et a la masse ajoute du fluide du fluide du fluide du flui
Une des propriétés ajoute du fluide du where Eij = 1/2(∂vi/∂xi) is the mean strain rate of the mean strain rate of the mean strain rate of the suspen
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$$
\phi \rho_p \frac{d_p \mathbf{u}_p}{dt} = \phi \rho_f \frac{d_f \mathbf{u}_f}{dt} + \nabla \cdot \sigma^p + \mathbf{f}^v + \phi(\rho_p - \rho_f) \mathbf{g}
$$
\n
$$
\rho_f \frac{d_f \mathbf{u}_f}{dt} = \nabla \cdot \tau^v - \mathbf{f}^v - \nabla p_f + \rho_f \mathbf{g}
$$
\nforce

\n
$$
\begin{array}{c}\n\bigcirc \bigcirc \\
\bigcirc \bigcirc \\
\mathbf{f}^v = -\frac{\phi \eta_f}{d^2} \lambda(\phi) (\mathbf{V}^p - \mathbf{V}) - \phi \nabla (\eta_f \nu(\phi) \dot{\gamma})\n\end{array}
$$
\nDarcy

\nResuspension

Continuum approach est en droit de negliger toutes les quantites de la continuum approach turbulence du fluide et a la masse ajoute du fluide et a la masse ajoute du fluide du fluide du fluide du flui
Une des propriétés ajoute du fluide du where Eij = 1/2(∂vi/∂xi) is the mean strain rate of the mean strain rate of the mean strain rate of the suspen
The suspen-to-suspen-to-suspen-to-suspen-to-suspen-to-suspen-to-suspen-to-suspen-to-suspen-to-suspen-to-suspen

$$
\phi \rho_p \frac{d_p \mathbf{u}_p}{dt} = \phi \rho_f \frac{d_f \mathbf{u}_f}{dt} + \nabla \cdot \sigma^p + \mathbf{f}^v + \phi (\rho_p - \rho_f) \mathbf{g}
$$

$$
\rho_f \frac{d_f \mathbf{u}_f}{dt} = \nabla \cdot \tau^v - \mathbf{f}^v - \nabla p_f + \rho_f \mathbf{g}
$$
force
$$
\Theta \qquad \Theta
$$

$$
\mathbf{f}^v = -\frac{\phi \eta_f}{d^2} \lambda(\phi) (\mathbf{V}^p - \mathbf{X}) - \phi \nabla (\eta_f \nu(\phi) \dot{\gamma})
$$

march is the liquid viscosity, d is the particle diameter and ¨γ = (2EijEiji)1/2EijEiji)1/2EijEiji) μ is the shear flow. The drag force dependiture dep particules sediment $\eta_f \dot{\gamma} \partial_y (\nu(\phi))$ expression [4] and Darcy Resuspension $\eta_f \dot{\gamma} \partial_y(\nu(\phi))$

dimensionless di↵usion coecient (Leighton & Acrivos 1987). The empirical relations 1 */*⁰ 1 */*⁰ tional to the tangent of the angle of repose. Another interesting finding is that it is also random loose packing $\phi_0 = 0.55$ and $\phi_0 = 0.55$ *^f*() = ¹ $\frac{u}{l} = \frac{9}{2} \Theta \int_0^{\phi_0} \left(\frac{1}{\phi} - \frac{1}{\phi_0} \right) \frac{D}{f \mu_r} \, \mathrm{d}\phi \approx 1.$ μ_r $\frac{h_u}{\sigma} = \frac{9}{2}\Theta \int^{\phi_0} \left(\frac{1}{\sigma} - \frac{1}{\sigma}\right) \frac{D}{\sigma} d\phi \approx 1.86\Theta,$ *d* $\partial \int_{0}^{\phi_0} \frac{1}{\phi_0}$ 0 *J M*
_ 0 $\approx 11.8\Theta.$ $\varphi_0 = 0.99$ $\frac{u}{d} = \frac{9}{2}\Theta$ \int^{ϕ_0} $\boldsymbol{0}$ $\left(\frac{1}{\phi} - \frac{1}{\phi_0}\right)$ ◆ *D* $\frac{\partial}{\partial \mu_r} d\phi \approx 1.86\Theta,$ *hl* $\frac{d}{d} = \frac{9}{2}\Theta$ \int^{ϕ_0} $\boldsymbol{0}$ 1 ϕ_0 *D* $\frac{\partial}{\partial \mu_r} d\phi \approx 11.8\Theta.$ $C = 4$ ripple formation under steady flow; more precisely, we show, on the basis of a simple $\phi_{0}=0.55$ $h \qquad \qquad$ $\int_{0}^{\phi_0}$ (1 1) $h_l = 9\omega_l \int^{\phi_0} 1 D$

for *f*(), *µr*() and *D*() proposed by Leighton & Acrivos (1986) are

random loose packing
\n
$$
\phi_0 = 0.55
$$
\n
$$
\frac{h_u}{d} = \frac{9}{2} \Theta \int_0^{\phi_0} \left(\frac{1}{\phi} - \frac{1}{\phi_0} \right) \frac{D}{f \mu_r} d\phi \approx 1.86\Theta,
$$
\n
$$
\frac{Q}{V_S d} = \frac{1}{V_S d} \int_{-h_l}^{h_u} \phi U d y = C \Theta^3,
$$

d ⇡ 1*.*86⇥*,* (B 4a)

$$
\frac{h_l}{d} = \frac{9}{2}\Theta \int_0^{\phi_0} \frac{1}{\phi_0} \frac{D}{f\mu_r} d\phi \approx 11.8\Theta.
$$
\n
$$
C = 4(\frac{9}{2})^3 \int_0^{\phi_0} \left\{ \frac{D}{f\mu_r} \int_{\phi}^{\phi_0} \frac{1}{\phi} \frac{D}{f\mu_r^2} d\phi \right\} d\phi \approx 7.5.
$$

no threshold u U II U no threshold

Figure 1. Schematic of a 2D Hagen-Poiseuille flow indicating the notation. Figure 1 Schematic of a $2D$ Hagen-Poiseville flow indicating the notation \sim (z). Where necessary with the subscripts \sim and 2 will distinguish between \sim particle layer that moved very rapidly. Finally, an additional flow instability was observed, a ripple type

 \bigcup is denoted by 2-D duct is denoted by 2-D duct is denoted by 2B. D

For uni-directional fully-developed laminar flows, the particle flux due to a gradient in concentration and a gradient in the shear stress τ is balanced by the particle flux due to gravity, hence p and p and p are sequence in this case of \mathcal{L}_{p} at low \mathcal{L}_{p}

 \sim the clear fluid and particle properties, respectively. Furthermore, the symbol g refers to the symbol

$$
\frac{2\phi a^2 \mathbf{g}\epsilon}{9v_1} f(\phi) + \dot{\gamma}(z) a^2 \hat{D} \frac{d\phi}{dz} + \dot{\gamma}(z) a^2 \tilde{D} \frac{1}{\tau} \frac{d\tau}{dz} = 0, \tag{1}
$$

The dotted line marks the position of the theoretically The dotted line marks the position of the theoretically
The dotted line marks the position of the theoretically The dotted line marks the position of the theoretical Figure 10. Measured interfacial velocity (\bullet) and comparison with the theoretical velocity profile (solid line). predicted interface, $\kappa = 0.004$, $\phi_s = 0.016$, $\rho_t = 963$ kg/m³, $v_1 = 2.86 \times 10^{-6} \text{ m}^2/\text{s}$ (water-ethanol mixture); $a = 185 \mu \text{m}$, $\rho_2 = 1043$ kg/m, (polystyrene beads).

Figure 11. Measured interfacial velocity $\left(\bullet \right)$ and comparison with the theoretical velocity profile (solid line). The dotted line marks the position of the theoretically predicted interface, $\kappa = 0.01$, $\phi_s = 0.065$, $\rho_1 = 963$ kg/m³, $v_1 = 2.86 \times 10^{-6}$ m²/s (water-ethanol mixture); $a = 185$ um, $\rho_2 = 1043$ kg/m, (polystyrene beads).

Reynolds of the particule

$$
Re = \frac{d^2}{\nu} \frac{\partial u}{\partial y}
$$

terminal velocity

$$
V_s = \frac{d^2}{18\eta} (\Delta \rho) g
$$

Reynolds associated= Galileo

Shields number

$$
Ga = \frac{(\Delta \rho)gd^3}{\eta^2}
$$

$$
\frac{\tau}{(\rho_p - \rho)gd} = \frac{Re}{Ga}
$$

Coulomb friction

 $\tau = \mu P$ $\tau=\eta$ ∂u $\frac{\partial y}{\partial D}$

 η μP $\overline{\partial u/\partial y}$

Ouiremi Aussillous Guazzelli

FIGURE 1. Sketch of a particle bed submitted to a Poiseuille (left) or a Couette (right) flow in a two dimensional channel.

Continuum approach Johunuum approach
Restierde pheee Dans ces equations les contraintes σpp sont liees aux forces directes Fpp entres les particules (comme les forces colloidales, les forces de collision ou les forces particule phase
particule phase particule phase

constraints exerted by the fluid constraints exerted by the particules conctrointe euc constraints exerted by the fluid e partic aints exerted by the particules IIes ≫ appropries pour decrire pour de constraints exerte appropries
appropries overted by the portioules

$$
\mathbf{F}^{pf} = n < \oint (\sigma^{f0} + p_f I) \cdot \mathbf{n} ds
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\n
$$
\mathbf{F}^{pf} = n < \oint (\sigma^{f0} + p_f I) \cdot \mathbf{n} ds
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\n
$$
\oint \phi_p \frac{d_p \mathbf{u}_p}{dt} + \nabla \cdot (\phi \rho_p < \mathbf{u}'_p \otimes \mathbf{u}'_p) = \nabla \cdot \sigma^{pp} + \mathbf{F}^{pf} - \phi \nabla p_f + \phi \rho_p \mathbf{g}
$$

σpp = n < fluid phase

les particules (comme les forces colloidales, les forces de collision ou les forces

p
p ⊗ u⁄
p ⊗ u′

^p [⊗] ^u′

 $\frac{1}{\sqrt{1-\frac{1}{2}}}\left(1-\frac{1}{2}\right)$

de contact) et sa definition generale est

fluid phase
$$
\overbrace{\nabla \cdot \tau^f \left(\mathbf{F}^{pf} - (1 - \phi) \nabla p_f \right)}^{(1 - \phi) \rho_f g}
$$

−(1 − φ)∇p^f + (1 − φ)ρfg (2)

$$
(1 - \phi)\rho_f \frac{d_f \mathbf{u}_f}{dt} + \nabla \cdot ((1 - \phi)\rho_f < \mathbf{u}'_f \otimes \mathbf{u}'_f) = \nabla \cdot \tau^f - \mathbf{F}^{pf} - (1 - \phi)\nabla p_f + (1 - \phi)\rho_f \mathbf{g}
$$

^p >) = ∇ · σpp + Fpf − φ∇p^f + φρpg (1)

 $p = \frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \right) \right) \right) \, dx}{\int_{0}^{\infty} \frac{1}{2} \left(\int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2$

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proposed a two phases problem (Jackson)

viscous law for the fluid

Darcy law for the interphase force

no resuspension

Coulomb friction for the dense phase

steady uniform flows

Ouiremi Aussillous Guazzelli

FIGURE 3. Numerical velocity profiles for the fluid $(+)$ and the particles (\times) in the case of particles of batch A in fluid 3 at $\phi_0 = 0.55$ and $\theta = 0.048$ (a), 0.094 (b), 0.62 (c) and analytical velocity profiles given in table 3 (solid line) and obtained by skipping region (II) (dashed line) (left). Blow-up of the profiles for the same conditions (right).

φ

² , (3.11)

viscosity is used in the numbers is used in the numbers of smaller Shields numbers α and α and α

d) at incipient motion. We recover that this threshold value is proporrandom loose packing, friction of the shields number of the bed-load term of the Shields $\phi_0 = 0.55$ µ = 0.43 as deduced by Cassar, Nicolas & Pouliquen (2005) for glass spherical particles $ac = \partial \bar{p}^f$, $\phi_0 \sim \mu^{\phi}$ This calculation also yields the bed-load thickness, h \sim thickness, h \sim the particle and fluid fluid flow- $\phi_0 = 0.55$ and $\phi_0 = 0.55$ and $\phi_0 = 0.55$ $\phi_0 = 0.55$ (solid line), versus analytical distribution $u = 0.43$ its previously mentioned hypotheses. Nonetheless, considering that the critical Shields incipient motion $\theta^c =$ $\partial \bar p^f$ $\frac{\partial P}{\partial \bar{x}} + \mu$ ϕ_0 $\frac{\partial}{\partial x}\approx\mu$ $\theta^c = \frac{\partial \bar{p}^f}{\partial \bar{z}} + \mu \frac{\phi_0}{2} \approx \mu \frac{\phi_0}{2},$ $\frac{2}{2}$, \bigcap rition \bigcap Choor **Critical Shear** and it is the angle of repose. Another interesting finding is that it is also considered in the set of α $\theta^c=0.12$ $\mathbf{v} = \mathbf{0.12}$ as deduced by Cassar, Nicolas spherical particles spherical pa $T_{r \cap \Omega} \cap T_{r \cap \Omega}$ Transport law $l=0.$ $\overline{3}$ \overline{a} \mathcal{O} at incipient motion. We recover that the this threshold value is proporproportional to the particle volume fraction in the bulk of the b \blacksquare cal shear, \blacksquare $t_{\text{rel}} = t_{\text{rel}}$ $\mu = 0.43$ random loose nacking triction random ioose packing, triction $\frac{1}{2}$ \mathcal{L} $\frac{0}{2} \approx \mu \frac{1}{2},$ Shindai Showing that the persuadies that the persuadies of the motion of the granular flow by the granular structure flow by the Transport law

its previously mentioned hypotheses. Nonetheless, considering that the critical Shields that the critical Shields

$$
q_p / \frac{\Delta \rho g d^3}{\eta_e} = \phi_0 \frac{\theta^c}{12} \left[\frac{\theta}{2\theta^c} \left(\frac{\theta^2}{\theta^c^2} + 1 \right) - \frac{1}{5} \right],
$$

 ${\sf Tr}$ angnort law at gmall Shieldge is \mathbb{R} $\frac{1}{2}$ Transport law at small Shields

 $\overline{}$

$$
q_p / \frac{\Delta \rho g d^3}{\eta_e} = \phi_0 \left(\frac{\mu_s}{\mu_2}\right)^2 \frac{\theta^c}{24} \left(\frac{\theta}{\theta^c}\right)^3.
$$

(II) (dashed line) (left). Blow-up of the profiles for the same conditions (right). FIGURE 3. Numerical velocity profiles for the fluid $(+)$ and the particles (\times) in the case of particles of batch A in fluid 3 at $\phi_0 = 0.55$ and $\theta = 0.048$ (a), 0.094 (b), 0.62 (c) and analytical velocity profiles given in table 3 (solid line) and obtained by skipping region

Numerical resolution

result: a critical shear a law of sediment transport Ouriemi Aussillous Guazzelli ODE

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Ouriemi Aussillous Guazzelli eq. α of the flow. In the regularisation and α is the magnitude α superior α small α $\frac{1}{2}$ solved in finite elements, homogenous in *x*

the set of governing equations for the two-phase problem. servation of mass, phase volume onservation of mass, phase volume fraction ! ! " of the two-phase model. In the two-fluid model the unknowns are \sim versus and pressure in each phase (fluid and particles) knowns that the number of extensions the number of the number of \mathcal{L}_i hods Appl. Mech. Engrg. 199 (2010) 439–449 curistivation un mass, phase Ω luma fra 2.1.2.1. Interaction term. Force for the average force for the average force $\frac{1}{2}$ CPU time the comparison is not known a priori. Actually, the reguwhere \mathcal{M} is designed by the volume fraction of the fluid phase. The parass, phase volume traction has the same form: rilass, phase volume fraction. conservation of mass, phase volume fraction $\overline{\mathsf{C}}$ $\begin{array}{ccc} \text{conservation of mass, phase} \ \text{conservation of mass} \end{array}$ tween phases. Using the Darcy law, the term n f ¹ put. Methods Appl. Mech. Engrg. 199 (2010) 439–449 $\,$ CONSETVA $\,$ ion of mass, phase volu aution

$$
\frac{\partial \epsilon}{\partial t} + \nabla \cdot \left(\epsilon \vec{u}^f \right) = 0, \qquad \frac{\partial \phi}{\partial t} + \nabla \cdot \left(\phi \vec{u}^p \right) = 0, \qquad \phi + \epsilon = 1.
$$

 $\cos \theta$ is the parameter the state $\sin \theta$ tion is phase continuity equation has the same form: onservatior velocity. The computational efficiency of the numerical models C fluid–particle interactions and stress tensor expressions. associated with the conservation of mo have validated the numerical model for Bingham fluid flows on conservation of momentum conservation of momentum method is easier than the Augmented Lagrangian one. Therefore we conservation technique to deal with the yields $\frac{1}{2}$ \sim Ouriemi et al. [26] for the flow of a Newtonian fluid over a granular give concluding remarks in Section 5. ticulate phase continuity equation \mathcal{C} continuity equation \mathcal{C}

$$
\rho_f \left[\frac{\partial \epsilon \vec{u^f}}{\partial t} + \nabla \cdot \left(\epsilon \vec{u^f} \otimes \vec{u^f} \right) \right] = \nabla \cdot (\overline{\overline{\sigma^f}}) - n\vec{f} + \epsilon \rho_f \vec{g},
$$
\n
$$
\rho_p \left[\frac{\partial \phi \vec{u^p}}{\partial t} + \nabla \cdot \left(\phi \vec{u^p} \otimes \vec{u^p} \right) \right] = \nabla \cdot (\overline{\overline{\sigma^p}}) + n\vec{f} + \phi \rho_p \vec{g},
$$
\n
$$
\rho_p \left[\frac{\partial \phi \vec{u^p}}{\partial t} + \nabla \cdot \left(\phi \vec{u^p} \otimes \vec{u^p} \right) \right] = \nabla \cdot (\overline{\overline{\sigma^p}}) + n\vec{f} + \phi \rho_p \vec{g},
$$
\n
$$
\rho_p \left[\frac{\partial \phi \vec{u^p}}{\partial t} + \nabla \cdot \left(\phi \vec{u^p} \otimes \vec{u^p} \right) \right] = \nabla \cdot (\overline{\overline{\sigma^p}}) + n\vec{f} + \phi \rho_p \vec{g},
$$

 ι ι ψ \mathcal{H}^{\bullet} λ . ($= \phi \nabla \cdot (\overline{\sigma})$ $\frac{1}{\sqrt{2}}$ $\eta + nJ^{\dagger}$ jeneral buoyancy + all other contributions \vec{r} \vec{r} respectively. The averages represents the average representation of $f(t) = \varphi \vee \cdot (\vartheta^{j}) + t \iota$ $n\vec{f} = \phi \nabla \cdot (\overline{\overline{\sigma}f}) + n \vec{f}^1$ \vec{f} \vec{f} \vec{f} \vec{f} \vec{f} $\mathbf{y} \sim \mathbf{y} \cdot \mathbf{y}$ present in Section 4 the section 4 the section 4 the two-phase model to simulate model to simulate model to si ulate the bed-load transport in the bed-load transport in the bed-load transport in the configurations, and \vec{r} \mathbf{f} is cross-section and a circular cross-section ducts. Finally, we can consider \mathbf{f} is $\$ ibut $\frac{1}{2}$ $\frac{1}{\sqrt{2}}$ all (
-- \overrightarrow{f}
 \overrightarrow{f} all other contribu $\frac{1}{4}$ ال $\frac{1}{2}$ $y + a$ θ and other correlation nt INUL $\frac{1}{\sqrt{2}}$ $\frac{1}{2}$ 5 فران
كان general buoyancy + all other contributions zi d \vec{F} \vec{F} $\frac{2}{\sqrt{2}}$ \overline{A} typical value for \overline{B} ∞ ontributions autions **contains** $\bigcap_{n=1}^{\infty}$ Paroulow Carmon Kazany apoficiant \bigcup \bigcup deneral buovancy + all other $a_j = \varphi v \cdot (v^j) + u_j$
Darcy Law, Carman-Kozeny coefici m deneral buo $m\vec{f}$ fuid and particulate phases respectively. The averages represents the average \mathbf{f} force exerted by the fluid on the particles and σ is the gravity acceleration and σ

 \overrightarrow{u} \overrightarrow{u} \overrightarrow{f} \overrightarrow{f} $m\vec{f}^1 = \eta \frac{\epsilon^2}{K} \left(\vec{u}^f - \vec{u}^p \right), \qquad K = \frac{\epsilon^3 d^2}{k c^2 (1 - \epsilon)^2}$ eration vector. TCY Law, Garman-KOZENY COENCIENT $\mu_J = \eta \frac{1}{K} (u - u)$, $\frac{1}{K} \left(\frac{1}{K} \frac{1}{K} \right)$ fluid–particle interactions and stress tensor expressions and stress tensor expressions. \bigcirc \bigcirc \bigcirc Darcy Law. Carman-Kozeny coeficient τ^2 ϵ^2 \rightarrow \rightarrow ϵ^3 ϵ^3 $nf^1 = \eta \frac{C}{K} \left(u^f - u^p \right), \qquad K = \frac{C u}{k \pi (1 - \epsilon)^2}$ $\overrightarrow{ }$
-1 $=\eta$ ϵ^2 $\frac{C}{K}$ $\left(u^f \right)$ \rightarrow $\left(\vec{u}^f - \vec{u}^p\right)$, $K = \frac{\epsilon^3 d^2}{k c^2 (1 - \epsilon)^2}$ coefficient $k_{CK} (1 - \epsilon)$ $\overline{2}$ the bed-load transport in laminar shearing flows. It is restricted to where r^f and r^p represent the stress tensor associated with the Darcy Law, Carman-Kozeny coeficient $\begin{array}{ccc} \hline \rightarrow & \\ \hline \end{array}$ $\eta_f = \eta_f$ yond the scope of this paper. We have considered two formulations ϵ^2 \rightarrow $\epsilon^3 d^2$ $f(w)$, $\mathbf{r} = \frac{1}{k_{cv}(1-\epsilon)^2}$ t ϵ^2 $(\vec{r}$ \vec{r} \vec{r} \vec{r} $\epsilon^3 d^2$ $n_j = \eta \frac{\mu}{K} (u^p - u^p), \qquad \kappa = \frac{1}{k_{CK}(1-\epsilon)^2}$ ev Law. Carman-Kozeny coeficient $N = \frac{N}{2}$ $\epsilon^3 d^2$ I þessum í þessum í
Í þessum í $\mu = \eta \frac{1}{K} (u - u)$, $\kappa - u$ S d^2 r $\frac{1}{2}$ $J₁$ $T-U^{\prime}$, $N=\frac{1}{k_{\text{eq}}(1-\epsilon)^2}$ \mathcal{L} solid friction in which the extra stress is proportional to \mathcal{L} $m \vec{f} = m \frac{\epsilon^2}{\omega} \left(\vec{f} - \vec{g} \right)$ $K = \frac{\epsilon^3 d^2}{\omega}$ $\mathcal{L}(K(T-\epsilon))$ $m f^* = \eta \frac{d}{d\mathcal{K}} \left(\frac{u^2 - u^2}{\mathcal{K}} \right),$ ー III
2 - 2 ^d³ ^ð¹ ^þ ^Rq^Þ D u^m \mathcal{L} $n\vec{f}^1 = \eta \frac{\epsilon^2}{K} \left(\vec{u^f} - \vec{u^p} \right), \qquad \qquad K = \frac{\epsilon}{k_{CK}}$ knowns than the number of equations then closure relations then closure relations \mathcal{F}

iuid stress tensor, effective viscosity Ouriemi et al. [26] for the flow of a Newtonian fluid over a granular bed in a two-dimensional configuration. After the validation \mathbf{I} 7
2
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 \overline{C} \overline{C} IS א וש
-! Fluid stress tensor, effective viscosity F_{fluid} otropo toppor offootive FIUID STRESS TENSOL, ETTECTIVE VISCOSITY of the two-fluid stress terms and the two-fluid model the unknowns are unknowns and unknown Γ eration vector. The set of partial differential equation of $\frac{1}{2}$ Fluid stress tensor, effective viscosity Fluid stress tensor, effective viscosity vertical upward direction, the vertical vector of the vertical vector of the k phase and the k ream-wise of the stream-wise of the later of the late vertical upward controlling and vertical upward vector of the control of the control of the k phase and the k phase and the control of the control Fluid stress tensor, effective vi $\mathsf{0}\mathsf{S}\mathsf{i}\mathsf{t}\mathsf{V}$ in tensorial form for $\mathsf{I}\mathsf{S}\mathsf{I}\mathsf{c}\mathsf{I}\mathsf{S}\mathsf{I}$ exerted by the particles can be determined in the particles can be determined in the particles can be decompos

$$
\overline{\overline{\sigma^f}} = -p^f \overline{\overline{I}} + \overline{\overline{v^f}} = -p^f \overline{\overline{I}} + \eta_e \left(\nabla \overrightarrow{u^m} + (\nabla \overrightarrow{u^m})^T \right), \quad \eta_e = \eta (1 + 5\phi/2)
$$

Particular by strong theory

annown σ stress terrsor, μ moder contributions. The first oriented buoyance and σ force all the second one gather ρ bucuo tunuu, pinnuu $\mathsf{str}\mathsf{f}$,00 iuin
-Particule stress tensor, µ model 2.1.1. Governing equations σ been chosen to been chosen to be applied to the concentrated situation: associated with both formulations is investigated in terms of accurations in terms of accurations \mathcal{L} the velocities and particular and particular and particles and particles $\mathsf{Part}(\mathsf{rule})$ its cartesian components are respectively denoted by P_2 Particule stress tensor, µ model ϵ and ϵ and ϵ and ϵ are ϵ and ϵ and ϵ are ϵ and ϵ are ϵ and ϵ are ϵ and ϵ and ϵ and ϵ are ϵ and ϵ and ϵ and ϵ are ϵ and ϵ and ϵ and ϵ and ϵ and ϵ Γ a viscous fluid fl tions reduced to the viscous drag force r

$$
\overline{\overline{\sigma^{p}}}=-p^{p}\bar{\overline{I}}+\overline{\overline{\tau^{p}}}\qquad\overline{\overline{\tau^{p}}}= \eta_{p}(\|\overline{\overline{\dot{\gamma}^{p}}}\|,p^{p})\overline{\overline{\dot{\gamma}^{p}}},\qquad \eta_{p}(\|\overline{\overline{\dot{\gamma}^{p}}}\|,p^{p})=\frac{\mu p^{p}}{\|\overline{\overline{\dot{\gamma}^{p}}}\|},\\ \overline{\overline{\dot{\gamma}^{p}}}=\nabla\overrightarrow{u^{p}}+(\nabla\overrightarrow{u^{p}})^{T}
$$

ever, let us mention that the implementation of the implementation of the regularization \bigcap we choose the regularisation technique to deal with the yield stress in our two-phase flow model. This method is advantageous model. This metho for its simplicity but one must be careful of the induced creeping \mathcal{L}_{max} flow in the \sim In this paper we present a three-dimensional finite element method (FEM) model of the two-phase incomparation \mathcal{L}_1 for both presented by \mathcal{L} \bigcap is to propose a numerical model able to propose a numerical model able to propose a numerical model as \bigcap the bed-load transport in laminar shearing flows. It is restricted to the cases where the granular bed does not change its shape in the granular bed does not change its shape in the \sim course of time, consequently ripped and dunnes formation are be- \sim 0.000 \sim \cup \cup \mathcal{P}_p and \mathcal{P}_p and $\overline{\mathcal{E}}$ have valid in the numerical model for L the test cases. We have compared the numerical model with an \sim analytical solution for the fluid between two solution for \mathcal{O}_1 \bigcup have also compared our model with \mathcal{P} $B_{\rm eff}$ in a square lide-driven cavity. Then we have validated cavity. Then we have validated α $\left(\begin{array}{ccc} \n\end{array} \right)$ and $\left[\begin{array}{cc} \n\end{array} \right]$ and $\left[\$ analytical solution for the fluid between two of a Bingham fluid between two of a Bingham fluid between two o infinite parallel planes. We have also compared our model with \sim \sim 100 σ mitsoulis and \sim 100 σ \bigcup Ouriemi et al. [26] for the flow of a Newtonian fluid over a granular $\frac{1}{\sqrt{2}}$ in the yielded regions that are given using regions $\frac{1}{\sqrt{2}}$ $\left(\begin{array}{ccc} \end{array} \right)$ \qquad $\$ \bigcap $f(x) = \bigcup$ and $f(x) = \alpha \nabla$ \sim 0 \sim \sim the paper we present a three-dimensional finite element and finite element a three-dimensional finite element and \sim \bigcup_{α} model of the two-phase incomparison of the two-phase incomparison model in \mathcal{L} for both presented by n \bigcirc concern is to propose a numerical model absolution \bigcirc \sim bed-log \sim bed-log \sim $\begin{pmatrix} 1 & 1 & 1 \ 2 & 2 & 1 \end{pmatrix}$ of the two-phase flow model flow model for p $\frac{1}{2}$ $2.1.1$ $\frac{1}{2}$ 2.1 \bigcap In this paper we present a three-dimensional finite element a three-dimensional f method (FEM) model of the two-phase incompressible flow model of the two-phase incompression \mathcal{L} for bed-load transport presented by Ouriemi et al. [26]. Our first concern is to propose a numerical model able to predict accurately \sim 1 the cases where the granular bed does not change its shape in the granular bed does not change its shape in the \sim course of time, consequently ripples and dunes formation are be- \sim scope of this paper. We have considered two formulations \sim of the two-fluid model. In the two-fluid model. In the unknowns are \mathcal{L} the velocities and pressure in each phase (fluid and particles) and particles (fluid and particles) and particles @^t ^þ ^r % ! uf 4 qp $\overline{}$ @^t ^þ ^r % / up ! ! " ² 7 eration vector.

and do not allow to conclude on the most efficient method. However, the most efficient method. However, \mathcal{L}_max

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\n
$$
\overline{\overline{v}^p} = \eta_p(|\overline{\overline{y}^p}|, p^p) \overline{\overline{y}^p},
$$
\nwith
\n
$$
\eta_p(|\overline{\overline{y}^p}|, p^p) = \frac{\mu p^p}{\|\overline{\overline{y}^p}\|},
$$
\n
$$
Ga = d^3 \rho_f \Delta \rho g / \eta^2
$$
\n
$$
Bn = \tau_0 H / \eta U
$$
\n
$$
\nabla \cdot (u^{\overline{i}^n}) = 0
$$
\n
$$
\nabla \cdot (u^{\overline{i}^n}) = 0
$$
\n
$$
G a^{\frac{12}{3} \cdot \frac{D \cdot u^{\overline{i}^n}}{D \cdot \overline{v}} = -\nabla p^f + \nabla \cdot \left(\frac{\eta_e}{\eta} \left(\nabla u^{\overline{i}^n} + \nabla u^{\overline{j}^n} \right) \right) - \frac{\mu^2}{K} \left(u^{\overline{i}^n} - u^{\overline{i}^p} \right) + \frac{\rho_f g}{\Delta \rho |\overline{g}|}
$$
\n
$$
G a^{\frac{H^2}{d^3} \cdot \overline{B \rho}} = -\nabla p^f + \nabla \cdot \left(\frac{\eta_e}{\eta} \left(\nabla u^{\overline{i}^n} + \nabla u^{\overline{j}^n} \right) \right) - \phi \nabla p^f
$$
\n
$$
+ \phi \nabla \cdot \left(\frac{\eta_e}{\eta} \left(\nabla u^{\overline{i}^n} + \nabla u^{\overline{j}^n} \right) \right) + \frac{(1 - \phi)H^2}{K} \left(u^{\overline{i}^n} - u^{\overline{i}^p} \right) + \frac{\phi \overline{g}}{|\overline{g}|}
$$

(a) Two-fluid model

Fig. 8. Comparison of the longitudinal velocity profiles for the flow of a Newtonian fluid over a granular bed between two infinite parallel planes obtained by numerica simulations (two-fluid model) with the analytical solution of Ouriemi et al. [26].

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Fig. 12. Velocity profile obtained by numerical simulations with the two-fluid model for the square cross-section duct ($6 \times 20 \times 40$). The fluid phase velocity is in blue and the particulate phase velocity is in red. An offset of 10^{-3} has been added to the velocity of the particulate phase (up) to make it visible. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Continuum approach Johunuum approach
Restierde pheee Dans ces equations les contraintes σpp sont liees aux forces directes Fpp entres les particules (comme les forces colloidales, les forces de collision ou les forces particule phase
particule phase particule phase

constraints exerted by the fluid constraints exerted by the particules conctrointe euc constraints exerted by the fluid e partic aints exerted by the particules IIes ≫ appropries pour decrire pour de constraints exerte appropries
appropries overted by the portioules

$$
\mathbf{F}^{pf} = n < \oint (\sigma^{f0} + p_f I) \cdot \mathbf{n} ds
$$
\n
$$
\mathbf{F}^{pf} = n < \oint (\sigma^{f0} + p_f I) \cdot \mathbf{n} ds
$$
\n
$$
\oint \phi_p \frac{d_p \mathbf{u}_p}{dt} + \nabla \cdot (\phi \rho_p < \mathbf{u}'_p \otimes \mathbf{u}'_p) = \nabla \cdot \sigma^{pp} + \mathbf{F}^{pf} - \phi \nabla p_f + \phi \rho_p \mathbf{g}
$$

σpp = n < fluid phase

les particules (comme les forces colloidales, les forces de collision ou les forces

p
p ⊗ u⁄
p ⊗ u′

^p [⊗] ^u′

 $\frac{1}{\sqrt{1-\frac{1}{2}}}\left(1-\frac{1}{2}\right)$

de contact) et sa definition generale est

fluid phase
$$
\overbrace{\nabla \cdot \tau^f \left(\mathbf{F}^{pf} - (1 - \phi) \nabla p_f \right)}^{(1 - \phi) \rho_f g}
$$

−(1 − φ)∇p^f + (1 − φ)ρfg (2)

$$
(1 - \phi)\rho_f \frac{d_f \mathbf{u}_f}{dt} + \nabla \cdot ((1 - \phi)\rho_f < \mathbf{u}'_f \otimes \mathbf{u}'_f) = \nabla \cdot \tau^f - \mathbf{F}^{pf} - (1 - \phi)\nabla p_f + (1 - \phi)\rho_f \mathbf{g}
$$

^p >) = ∇ · σpp + Fpf − φ∇p^f + φρpg (1)

 $p = \frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \right) \right) \right) \, dx}{\int_{0}^{\infty} \frac{1}{2} \left(\int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \int_{0}^{\infty} \frac{1}{2} \left(\frac{1}{2$

free fall equilibrium

$$
\frac{4}{3}\pi(\Delta\rho)R^3g=6\pi\eta RV_s
$$

terminal velocity

$$
V_s = \frac{d^2}{18\eta} (\Delta \rho) g
$$

flux

$$
Q_s = \frac{d^3}{\eta} (\Delta \rho) g
$$

free fall equilibrium

terminal velocity

$$
V_s = \sqrt{\frac{(\Delta \rho)gd}{\rho C_D}}
$$

flux

$$
Q_s=\sqrt{\frac{(\Delta\rho)gd^3}{\rho}}
$$

R. Bakhtyar et al./Advances in Water Resources 33 (2010) 277–290 $cos in$ Water Pescurses 22 (2010) 277 200 tto in water hesoarces so (2010) 211–250 Bakhtyar et al./Advances in Water

> 2.1. Governing equations tively; \mathbf{t} the drag force is the resistance experience experience experienced by a grain moving in \mathbf{t} 2.1. Governing equations Governing equations \mathfrak{g} cy *ations*

Fluid phase hydrodynamics [5] id phase hydrodynamics [5] around the sediments. The sediments \mathbf{c}

$$
\frac{\partial [(1 - C)\rho_f]}{\partial t} + \nabla \cdot [(1 - C)\rho_f \cdot \vec{U}] = 0,
$$
\n(1)

$$
\frac{\partial [(1-C)\rho_f \vec{U}]}{\partial t} + \nabla \cdot [(1-C)\rho_f \vec{U} \vec{U}]
$$

= (1-C)\rho_f \vec{g} - \nabla \cdot [(1-C)P] + \nabla \cdot \vec{\vec{T}}_f - \vec{f}_{in}, (2)

Sediment phase hydrodynamics [5] iment phase hydrodynamics [5] s liment phase hydrodynamics $[5]$ $\phi(C_{\Omega})$

$$
\frac{\partial(C\rho_s)}{\partial t} + \nabla \cdot (C\rho_s \vec{V}) = 0,
$$
\n(3)

$$
\frac{\partial (\mathcal{C}\rho_s \vec{V})}{\partial t} + \nabla \cdot (\mathcal{C}\rho_s \vec{V} \vec{V}) = \mathcal{C}\rho_s \vec{g} - \nabla \cdot (\mathcal{C}P) + \nabla \cdot \vec{T}_s + \vec{f}_{in},
$$
(4)

Interphase forces [12,37]

 $T_{\rm eff}$

$$
\vec{f}_{in} = \frac{3C\rho_f}{4d}C_D(\vec{U}-\vec{V})|\vec{U}-\vec{V}| + \frac{3C\rho_f}{4d}C_L|\vec{U}-\vec{V}|\nabla(\vec{U}-\vec{V}) \cdot \vec{k}, \quad (5)
$$

$$
C_D = \left(\frac{24v}{d|\vec{U}-\vec{V}|}+2\right)(1-C)^{-9/2} \text{ and } C_L = \frac{4}{3}, \quad (6)
$$

$$
\overline{U'_i U'_j} = -v_t (U_{i,j} + U_{j,i}) + \frac{2}{3} \delta_{ij} k - \frac{2}{3} v_t \delta_{ij} U_{j,i},
$$

$$
\overline{C'U'_i} = \overline{C'V'_i} = -v_t \frac{\partial C}{\partial z},
$$

$$
v_t = C_d \frac{k^2}{\varepsilon},
$$

 \overline{C}

$$
\begin{array}{c}\n0 \\
0 \\
0\n\end{array}
$$

$$
\frac{Dk}{Dt} = \nabla \cdot \left(v + \frac{v_t}{\sigma_k} \right) \nabla k + P_r + \frac{\rho_s - \rho_f}{\rho_f} \vec{g} v_t \nabla C \cdot \vec{k} - \varepsilon - v_t F_d \nabla C \cdot \vec{k},
$$
\n
$$
\frac{D\varepsilon}{Dt} = \nabla \cdot \left(v + \frac{v_t}{\sigma_\varepsilon} \right) \nabla \varepsilon
$$
\n
$$
+ \frac{\varepsilon}{k} \left(C_{1\varepsilon} P_r + C_{2\varepsilon} \vec{g} v_t \frac{\rho_s - \rho_f}{\rho_f} \nabla C \cdot \vec{k} - C_{3\varepsilon} \varepsilon - v_t F_d \nabla C \cdot \vec{k} \right),
$$
\n
$$
F_d = \frac{C}{1 - C} \frac{3C_D}{4d} |\vec{U} - \vec{V}|^2,
$$
\n
$$
C_{1\varepsilon} = 1.44, \quad C_{2\varepsilon} = 1.92, \quad C_{3\varepsilon} = 1.2,
$$
\n
$$
\text{Closure of particle stresses [3,11]}
$$
\n
$$
T_{xz} = \frac{6v\rho_f}{5[(C_{\rm m}/C)^{\frac{1}{3}} - 1]^2} \frac{\partial V_x}{\partial z},
$$
\n
$$
(1.41)
$$

$$
5[(C_m/C)^5 - 1]^2 \quad \text{or} \quad \text{or} \quad T_{zz} = \frac{T_{xz}}{\tan \varphi}, \tag{1}
$$

more simple models but no shallow water Over simplification: considering only the Stokes velocity

Over simplification: considering only the fall velocity

Over simplification: considering only the Stokes velocity

free fall equilibrium

$$
\frac{4}{3}\pi(\Delta\rho)R^3g=6\pi\eta RV_s
$$

terminal velocity

$$
V_s = \frac{d^2}{18\eta} (\Delta \rho) g
$$

characteristic flux

$$
Q_s = \frac{d^3}{\eta} (\Delta \rho) g
$$

 $n:$ con dfu^f lC r ing only the Stakes velocity Over simplification: considering only the Stokes velocity

 $n:$ con dfu^f lC r ing only the Stakes velocity Over simplification: considering only the Stokes velocity

 $n:$ con dfu^f lC r ing only the Stakes velocity Over simplification: considering only the Stokes velocity

$$
\phi \rho_p \frac{d_p \mathbf{u}_p}{dt} = \phi \rho_f \frac{d_f \mathbf{u}_f}{dt} + \nabla \cdot \sigma^p + \mathbf{r} \cdot \phi (\rho_p - \rho_f) \mathbf{g}
$$
\n
$$
\rho_f \frac{d_f \mathbf{u}_f}{dt} = \nabla \tau_v - \nabla p_f + \rho_f (1 + \phi \frac{\rho_p - \rho_f}{\rho_f}) \mathbf{g}
$$
\n
$$
\text{dilute} \qquad \text{excess of mass}
$$
\n
$$
\mathbf{f} = -\frac{\phi \eta_f}{d^2} \lambda(\phi) (\mathbf{V}^{\mathbf{p}} - \mathbf{V})
$$

Darcy

$$
\overrightarrow{u_p} \simeq \overrightarrow{u}_f + \overrightarrow{V}_f + \dots
$$

one fluid

one fluid

Velocity of the sediments:

 $u_p = u$ $v_p = v$ Convection one fluid

Velocity of the sediments:

 $u_p = u$ $v_p = v - V_f$ **Sedimentation**
Velocity of the sediments:

$$
u_p = u - D \frac{\partial c}{\partial x}
$$

$$
v_p = v - V_f - D \frac{\partial c}{\partial y}
$$

Diffusion

Mass conservation of the sediments:

local form;

$$
\frac{\partial cu}{\partial x} + \frac{\partial c(v - V_f)}{\partial y} = \frac{\partial c}{\partial x}D\frac{\partial c}{\partial x} + \frac{\partial}{\partial y}D\frac{\partial c}{\partial y}
$$

Mass conservation of the sediments:

local form;

$$
\frac{\partial cu}{\partial x} + \frac{\partial c(v - V_f)}{\partial y} = \frac{\partial c}{\partial x}D\frac{\partial c}{\partial x} + \frac{\partial}{\partial y}D\frac{\partial c}{\partial y}
$$

integral form: $\int_0^\infty c u dy = q$, ...

$$
\frac{\partial q}{\partial x} + c(x,0)(V_f) = -D\frac{\partial c}{\partial y}(x,0) \qquad \frac{\partial f}{\partial t} = c(x,0)(V_f) + D\frac{\partial c}{\partial y}(x,0)
$$

Mass conservation of the sediments:

local form;

$$
\frac{\partial cu}{\partial x} + \frac{\partial c(v - V_f)}{\partial y} = \frac{\partial c}{\partial x}D\frac{\partial c}{\partial x} + \frac{\partial}{\partial y}D\frac{\partial c}{\partial y}
$$

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$$
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$$

$$
\text{if }\qquad \frac{\partial \tilde{u}}{\partial \tilde{y}}|_0 > \tau_s \qquad \text{then} \qquad -\frac{\partial \tilde{c}}{\partial \tilde{y}}|_0 = \beta \big(\frac{\partial \tilde{u}}{\partial \tilde{y}}|_0\big)^a \big(\frac{\partial \tilde{u}}{\partial \tilde{y}}|_0 - \tau_s\big)^b, \qquad \text{else} \qquad -\frac{\partial \tilde{c}}{\partial \tilde{y}}|_0 = 0.
$$

$$
\frac{\partial f}{\partial t} = -\frac{\partial q}{\partial x}.
$$

Brivois 2005/ Lagrée 2000, 2003

Mass conservation of the sediments:

local form;

$$
\frac{\partial cu}{\partial x} + \frac{\partial c(v - V_f)}{\partial y} = \frac{\partial c}{\partial x}D\frac{\partial c}{\partial x} + \frac{\partial}{\partial y}D\frac{\partial c}{\partial y}
$$

integral form: $\int_0^\infty c u dy = q$, ...

Mass conservation of the sediments:

local form;

$$
\frac{\partial cu}{\partial x} + \frac{\partial c(v - V_f)}{\partial y} = \frac{\partial c}{\partial x}D\frac{\partial c}{\partial x} + \frac{\partial}{\partial y}D\frac{\partial c}{\partial y}
$$

integral form: $\int_0^\infty c u dy = q$, ...

$$
\frac{\partial q}{\partial x} + qV_f/H = \beta \left(\frac{\partial \tilde{u}}{\partial \tilde{y}}|_{0}\right)^{a} \left(\frac{\partial \tilde{u}}{\partial \tilde{y}}|_{0} - \tau_s\right)^{b} \qquad \frac{\partial f}{\partial t} = -\frac{\partial q}{\partial x}.
$$

$$
l_{sat}\frac{\partial q}{\partial x} + q = q_{sat}
$$

$$
q_{sat} = E(\tau - \tau_s)_{+}
$$

$$
l_s \frac{\partial q}{\partial x} + q = q_s \qquad \frac{\partial f}{\partial t} = -\frac{\partial q}{\partial x}
$$

$$
q_s = E(\tau - \tau_s)_+
$$

 $A(n)$ or $A(n)$ independent of the spanwise coordinate $A(n)$ $\rho(\mathbf{x}) = \rho_f[1 + \gamma c(\mathbf{x})],$ where $\gamma = (\rho_p - \rho_f)/\rho_f$ $T_{\rm eff}$ monodisperse particles are advected by the fluid, while settling with a constant \sim $s(x) = s$ in denotes the stress $s(t) = s$ $p(x) - p_f[1 - \gamma c(x)]$, where $\gamma - (\rho_p - \rho_f)/\rho_f$.

Deep-water sediment wave formation: linear stability analysis of coupled flow/bed interaction \overline{a} recent \overline{a} recent \overline{a} Deep-water sediment wave formation: linear stability analysis o comprise the government of the government of the coupled for the coupled for the coupled flow/sediment bed dynamics. At the coupled for the co the interface, a no-slip condition is imposed for the horizontal u-velocity component, ∂t anaiysis or coupleu

Lesshafft Hall Meiburg Kneller JFM11 $2.3.1$

Mixing and dissipation in particle-driven gravity currents iviixing and dissipation in particle-driven gravity currents
Neckel Härtel Kleiser Meiburg JFM 05 The interstitial void fraction in the sediment bed is not taken into account; assuming into the interior of the interior of the fluid is modelled as a diffusive flux (Parker 1978; Blanchetter 1978; advances or recedes, The governing equations are rendered dimensionless with respect to a diffusive length in particle-driven gravity currents
. (2008), is not renormalized in the present framework. In the present framework, it is not renormalized in the present framework.

blem represents the elementary archetype of a gravity current. The specific case of local currents, where the local currents, where density currents, where density currents, where α Neckel Härtel Kleiser Meiburg JFM 05

one fluid

FIGURE 3 – Ecoulements gravitaire sur une pente de 15^o, $t = 0, 1, 2, 3, 4$, 5 et 10. $V_f \neq 0$

FIGURE 5 – Ecoulements gravitaire sur une pente de 15^o comparaison au temps $t = 3$. $V_f \neq 0$ en haut et $V_f = 0$ en bas, on constate que la vitesse de sédimentation V_f fait retomber le panache. pour $V_f \neq 0$ le front de l'avalanche est plus ramassé.

 $\mathcal{D}u + v = u = u_c(x) - u_c(x) + \cdots$ $u_e(x)$ + $\overline{2\tilde{x}}$ $\mu = \mu_e(x)$ $y = u + v = v$ et $u = u + v = u$ if $u = u_e(x) - u_e(x) + v_e(x)$ interaction $u +$ $v = v$ et u $\mu + \nu$ $\partial \bar{\kappa}$ $\hat{\mathcal{W}}$ ∂ *i* $\bar{\mathfrak{c}}$ *y*˜ $\vec{\mathsf{p}}$ $\tilde{u}(\bar{x}, \tilde{y} = 0) = 0, \tilde{v}(\bar{x}, \tilde{y} = 0) = 0$ = let lim $\tilde{u}(\bar{x},\tilde{y})=\bar{u}_e(\bar{x})$ water g $\frac{\tilde{y} \rightarrow \infty}{\ }$ δ $f(x)$ $\overline{0}$ and $\overline{0}$ intervals with $x=L$
and $\overline{0}$ intervals par la solution de Blasius. wall L **Figure 1.** Sketch of the flow, before $\bar{x} = 1$ the bottom is not erodable. **Figure 1.** Une vue de l'écoulement, avant $\bar{x} = 1$ le sol est fixe. ideal fluid $\omega e^{i\omega} = \omega_e e^{i\omega} = 1 - \omega_e$ Equations de transport Equations de transport $\frac{\partial}{\partial x}$ $\frac{\partial}{\partial y}$ $\frac{\partial}{\partial y}$ $\frac{\partial}{\partial y}$ $\frac{\partial}{\partial y}$ $\bar{u}_e(\bar{x}) + \frac{\partial^2}{\partial \tilde{x}^2}$ ∂ ∂ ∂ ∂ $\tilde{u} = \bar{u}_e(\bar{x}) \frac{\mathrm{d}}{\mathrm{d}\bar{z}}$ \tilde{u} + $\tilde{v}=0$ et \tilde{u} $\tilde{u}+\tilde{v}$ $\frac{\widetilde{\sigma}}{\partial \widetilde{y}^2}\tilde{u}$ $\partial \bar x$ $\partial \tilde y$ $\partial \bar x$ $\partial \tilde y$ $\mathrm{d}\bar{x}$ boundary layer $\tilde{u}(\bar{x},\tilde{y}=0)=0,\tilde{v}(\bar{x},\tilde{y}=0)=0 \quad \text{et} \quad \text{lim}$ $\tilde{u}(\bar{x},\tilde{y})=\bar{u}_e(\bar{x})$ $u(x, y = 0) = 0, v(x, y = 0) = 0$ et $\lim_{\tilde{y} \to \infty} u(x, y)$ $\tilde{y} \rightarrow \infty$ $\mathcal{G} \cap \infty$ ∂_{ε}^2 $\tilde{u} \frac{\partial}{\partial x}$ \tilde{x} $\frac{\partial}{\partial \vec{x}}\frac{\partial}{\partial \vec{x}}\tilde{c} + \tilde{V}$ ∂^2 \vec{u} $\frac{\partial}{\partial t}$ $\tilde{c}+\tilde{V}(\tilde{u})\frac{\partial}{\partial \tilde{u}}\tilde{\tilde{v}}_f \neq \frac{\partial}{\partial \tilde{u}}$ $\tilde{u} \stackrel{\overrightarrow{O}}{\longrightarrow} \tilde{x} + \tilde{\mathcal{C}}_{\tilde{v}} \tilde{z} \stackrel{\overrightarrow{O}}{\longrightarrow} \tilde{w} \stackrel{\overrightarrow{O}}{\longrightarrow} \tilde{w} \stackrel{\overrightarrow{O}^2}{\longrightarrow} \tilde{w} \stackrel{\overrightarrow{O}^2}{\longrightarrow} \tilde{w} \stackrel{\overrightarrow{O}^2}{\longrightarrow} \tilde{w}$ \widetilde{C} $\widetilde{C} = \frac{1}{\sqrt{2}} \widetilde{C}^1$ transport $\tilde{u} \frac{\partial}{\partial \tilde{x}} \tilde{v} \frac{\partial}{\partial \tilde{v}} \tilde{v} + V(\tilde{u}) \frac{\partial}{\partial \tilde{v}} \tilde{v} + S_c \tilde{c} - \tilde{S}_c \tilde{v}$ 2 $\frac{\widetilde{C}}{\partial \widetilde{y}^2}\widetilde{c}$ $\overrightarrow{y}^{\partial}_{\overrightarrow{\cdot}}$ $\begin{array}{c} \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial}{\partial t} \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial}{\partial t} \frac{\partial}{\partial t$ \widetilde{c} $\widetilde{\mathscr{L}}\widetilde{g}$ $\widetilde{g}^{-\widetilde{\mathscr{C}}}_{c}$ $\partial \vec{x}$ $\partial \vec{y}$ $\partial\!\!\!/\,\overline{\!\!\vec{\mathcal{U}}}$ $\partial \vec{y}$ $\frac{\partial \widetilde{\mathcal{Y}}^{2C} }{\partial \widetilde{y}^{2}} \widetilde{c}$ manopont $\partial \bar{x}^{\circ}$ and $\partial \bar{y}^{\circ}$ is $\partial \tilde{y}^{\circ}$ if $\partial \tilde{y}^2$ $\partial \bar x$ $\partial \tilde y$ \bigwedge γ $\begin{array}{c} \hline \end{array}$ $\overline{\mathbf{r}}$ $\overline{}$ $\tilde{\mathfrak{A}}(\bar{\mathfrak{D}}) \leq \tilde{d}(\bar{x}) \leq \tilde{d}(\bar{x},\tilde{y}) \leq \tilde{e}(\bar{y},\tilde{y}) \leq \tilde{e}(\bar{x},\tilde{y}) \Rightarrow \mathcal{R}_{\tilde{y}} \geq 0 \Rightarrow \mathcal{R}_{\tilde{y}} \$ $\overline{1}$ $\int_{\Omega} \partial \tilde{u}$ ◆◆ ✓ ⇤*u*˜ $\overline{}$ $\partial\hat{\mathcal{C}}$ $\int \partial \tilde{u}$ $\overline{1}$ $\partial \tilde{\partial} \tilde{\psi} \big| \, \big/ \, \partial \tilde{\psi}$ \int $d(\tilde{x}) = 0$ \tilde{y} = $\tilde{e}(\bar{x}, \tilde{y} \tilde{c} \left(\bar{x}, \tilde{y}) \Rightarrow \mathbb{Q}_0\right) = 0$ $\begin{array}{c} \hline \end{array}$ I $\overline{}$ $\left| \right|$ $\overline{}$ $\overline{}$ \rightarrow \overline{a} $\equiv \beta$ H $\overline{}$ $\frac{\partial u}{\partial \tilde{u}}$ $\overline{}$ $\frac{\partial u}{\partial \gamma}$ s \overline{H} $\frac{1}{2}$ $\frac{1}{\sqrt{2\pi}}\left|\frac{1}{\sin\theta}\right|$ erosion $\hbox{\tt\;}\not{\!\! \rm\;}\, \beta$ H $\frac{1}{\sqrt{2}}$ $- \tau_s$ $\frac{\partial \mathcal{L}}{\partial \mathcal{L}} \left\{ \mathcal{L} \otimes \mathcal{L} \right\} = \frac{1}{2} \sum_{i=1}^n \left| \mathcal{L} \otimes \mathcal{L} \right| \mathcal{L} \left\{ \mathcal{L} \otimes \mathcal{L} \right\} = \frac{1}{2} \sum_{i=1}^n \left| \mathcal{L} \otimes \mathcal{L} \right| \mathcal{L} \left\{ \mathcal{L} \otimes \mathcal{L} \right\} = 0$ $\left| \cdot \right|$ Щ $\overline{}$ \mathbb{I} $\overline{}$ \int $\overline{}$ $\overline{}$ $\overline{}$ $\partial \tilde{y}$ $\delta \tilde{y}$ $\partial \tilde{y}$ $\partial \tilde y$ $\partial \widetilde{y}$ $\partial \widetilde{y}$ \vert_0 $\partial \tilde{y}\left| _{0}\right\rangle$ $\overline{\mathcal{A}}$ $\big]0$ $\langle \partial \tilde{y} | \partial$ \vert_0 $\langle \partial \tilde{y} | \phi \rangle$ \vert ⁰ $\frac{1}{6}$ $\frac{1}{6}$ ⇤*y*˜ ⇤*y*˜ ⇤*y*˜ $\frac{1}{\sqrt{2}}$ $\frac{1}{1}$ $\overline{1}$ \cup \cup \cup $\partial \hat{f}$ $\overline{\mathbf{r}}$ \mathbf{q} ^{\mathbf{r}} \overline{a} $\partial \tilde{c}$ $\tilde{\tau}$ $\overline{}$ $+$ $\tilde{V}_f \tilde{c}|_0$ $\frac{\partial J}{\partial \check{t}} = S_c^{-1}$ r
f $|0 \tV_f \tilde{c}|_0$ *J*
) *j =* $S_c^{-1} \left. \frac{\partial c}{\partial \tilde{y}} \right|_0$ $\frac{c}{\tilde{n}}$ Lagrée 00 $\overline{}$ update*c* \int \mathbf{r} \overline{a} ⁺ *^V*˜*^f ^c*˜*|*⁰ \overline{c} \overline{c} $\partial \tilde y$ \vert_0 *y*˜

Flow over a bump, ideal fluid / boundary layer interaction

$$
\alpha = 0.225
$$
, Fr = 0.6 , $\varepsilon = 500^{-\frac{1}{2}}$, $\beta = 0.8$, $\tau_s = 0.35$, Sc = 1 et $\tilde{V}_f = 1$

Lagrée 00

Flow over a bump, ideal fluid / boundary layer interaction

Lagrée 00

Figure 3. The dune shape $(\hat{f}(\bar{x},\tilde{t}))$ as a function of time $\check{t} = 0, 1, 2, 3, \ldots, 16, \infty.$ Figure 3. Évolution de la forme de la bosse en fonction du temps $\check{t} = 0, 1, 2, 3, \ldots, 16, \infty.$

dunes pour différentes valeurs de α .

Sculpting of an erodible body by flowing water

L. Ristroph, M. Moore, S. Childress, M. Shelley et J. Zhang

PNAS 12

Moore, Ristroph, Childress, Zhang, Shelley PoF 13 Self-similar evolution of a body eroding in a fluid flow

Fig. 1. Erosive sculpting of a clay sphere by flowing water. The flow speed is 46 cm/s and initial diameter 4.9 cm. (A) Removed after 70 min of erosio a water tunnel, the body has a conelike nose, angular ridges, and flattened back. (B and C) Photographs of 10-ms exposure reveal streaklines at 10 and 70 r The flow (left to right) conforms to the body before separating and forming a complex wake downstream. (D and E) Schematics show that the sharp feat at the nose and ridges are associated with the flow stagnation point (SP) and separation lines (SL), respectively.

FIG. 2. Visualizing the flow around a cylindrical body at different times in the erosion process. Streaklines are captured by 10 ms exposure time photographs of tracer particles illuminated by a laser sheet, and the initial diameter of the body is 3.6 cm. (a) Early in the process, $t = 5$ min, the incoming flow stagnates at the nose and conforms to the body until separating just upstream of the widest portion. The wake behind the body consists of a relatively slow and unsteady flow. (b) At $t = 55$ min, the body has formed a quasi-triangular shape, yet the flow structure is qualitatively similar. The flow stagnates at the nose and separates near the body's widest portion, in this case near the back corners of the triangular shape. (c) and (d) Flow schematics. Ω schematics.

FIG. 3. High-Re flow past a bluff body in two dimensions. (a) The fluid flow is comprised of an outer and boundary layer how, with the dashed curve mulcating the thickness of the boundary rayer. At the separation point, the boundary rayer detaches
and a wake is formed. The dotted curve represents the separating streamline. (b) Zoom into the profile inside the boundary layer approaches the outer tangential velocity $U(s)$ at a characteristic distance $\delta(s)$. Fluid shear stress is proportional to the slope of the velocity profile at the surface (darkened). flow, with the dashed curve indicating the thickness of the boundary layer. At the separation point, the boundary layer detaches

$\sum_{\text{time (min)}}$ $\frac{1}{\sqrt{2\pi}}\int d^3x\sqrt{\frac{1}{2}}\left(\frac{1}{2}\frac{1}{2}\right)\left(\frac{1}{2}\frac{1}{2}\right)\left(\frac{1}{2}\frac{1}{2}\right)\left(\frac{1}{2}\frac{1}{2}\right)\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)\left(\frac{1}{2}\frac{1}{2}\frac{1}{$ 0 30 60 90 115 time (min) *s* \sim selfsimilarity

ideal fluid $\nabla^2 \phi = 0, \quad \nabla \phi \cdot \hat{\mathbf{n}} = 0 \quad \text{on } \partial B$ and implies that I ideal fillid I in the potential streamlines. In the potential, the potential, the point of the potential streamlines of the postential streamlines of the potential, the point of the potenti

andary layer **and a** The outer flow has vanishes on the boundary, as indicated by the boundary, as indicated by the Neumann condition \downarrow Soundary layer

$$
u \frac{\partial u}{\partial s} + v \frac{\partial u}{\partial n} - v \frac{\partial^2 u}{\partial n^2} = U U',
$$

normalized arc length, s

$$
u \frac{\partial u}{\partial s} + v \frac{\partial u}{\partial n} - v \frac{\partial^2 u}{\partial n^2} = U U',
$$

$$
\frac{\partial u}{\partial s} + \frac{\partial v}{\partial n} = 0.
$$

erosion of the Pranch Pranch Pranch Prandtl PDEs to obtain an ordinary differential equation (ODE) for boundary differential equation (ODE) for boundary differential equation (ODE) for boundary differential equation (ODE) α *s* α α β sponding velocity components (*u*, v) of the inner flow. The inner flow a boundary α boundary The inner region consists of a thin layer surrounding the solid boundaries, as shown in $\mathcal{O}_\mathcal{S}$ effects of viscosity come to be effects of viscosity come to be effects of the effects

 $V_n = -C |\tau|$, \mathcal{L} $V_n = -C$ | τ |, $\mathbf{v}_n = \mathbf{v}_1$ content is subject to the terms at \mathbf{v}_n .

cf Brivois Bonelli Borghi 07 (en turbulent) erode. Brivois-Bonelli Borghi 07 (en turbulent) layer surroundtransverse to the flow; e.g., for the case of a circle, *a* is simply the radius. Boundary conditions for $BR₁$ In Sec. V, we will describe the results of the simulation in terms of dimensionless quantities,

ideal fluid
\n
$$
m = \alpha/(\pi - \alpha)
$$
,
\n $U(s) = c_0 s^m$

Boundary layer ⁼ *mc*² 0*s*²*m*−¹ \overline{a} ∙
∂2ψ
2⊎ $\frac{w}{2} + \frac{1}{2} (m +$ $\frac{1}{2}$ *n*³ *j j j* $f''' + \frac{1}{2}(m+1)ff'' - mf'^2 + m = 0$ $\,$ Introduceducing the similar similar similar strength in the strength of *n u*, and the strength strength in the strength of $\frac{1}{2}$ as *γγγραφούς του βρίσιου του περεί το στο ποι* στο *ξωμής το στο βρίσιο* του επιλεί το στο βρίσιο του επιλεί του
Γεγονότα το στο βρίσιο του επιλεί του επιλεί του προσπάθει το προσπάθει το προσπάθει το προσπάθει το προσπ 1 2 $(m + 1) ff'' - mf'^2 + m = 0.$ [∂]*n*∂*^s* [−] ∂ψ ∂*s* [∂]*n*² [−] ^ν ∂*n*³ $\frac{1}{\epsilon} \int_{0}^{1} \frac{-\delta(s)}{s^2} dx = \frac{1}{\epsilon^{2}}$, $\frac{1}{\epsilon^{2}}$, $\frac{1}{\epsilon^{$

erosion $V_n = -C |\tau|$, $\tau(s) = c_1 s^{(3m-1)/2}$. $\mathbf{f}(\mathbf{r})$ in the literature, 15 which sometimes incorporate additional parameters, 16 which sometimes incorporate additional parameters, 16 which sometimes incorporate additional parameters, 16 which sometimes incorp (*^m* ⁺ 1) *f f* ′′ [−] *m f* ′² ⁺ *^m* ⁼ ⁰. (25) \sim μ \sim μ ¹ $V_n = -C |\tau|$, $\frac{1}{1}$ ν*c*³ ⁰ *f* ′′(0), where *f* ′′(0) may be determined by the numerical $V_{\rm c} = -C |\tau|$ $\frac{1}{\sqrt{2}}$ is not not needed to understand how the shear stress depends on the wedge stress depends on the wedge $\frac{1}{\sqrt{2}}$ $\tau(s) = c_1 s^{(3m-1)/2}$. shear stress near the nose, *s* = 0. In Fig. 5(a), we illustrate the velocity field around a narrow wedge, (*^m* ⁺ 1) *f f* ′′ [−] *m f* ′² ⁺ *^m* ⁼ ⁰. (25) V $\tau(s) = c_1 s^{(3m-1)/2}$ \bullet

 $\overline{1}$

such as a power for the shear stress and/or a threshold shear stress below which material does not onsta The constant is given by *c*¹ = ρ ν*c*³ $constant$ erosion $m = 1$!!! ν*c*³ α $\text{constant erosion} \quad m = 1/3 \qquad \qquad \alpha = \pi/4$

for the stress is associated with a thin boundary $\alpha = \pi/4$ \int ² \int ¹ $\overline{14}$ ⁰ *f* ′′(0), where *f* ′′(0) may be determined by the numerical $\alpha - \pi/4$ $\alpha = \pi / 1$

FIG. 5. Falkner-Skan similarity solutions for flow past wedges. (a) Illustration of the outer and boundary-layer flow past a wedge with acute opening angle. The shear stress is highest near the nose as indicated by the surface slopes of the velocity profiles. This causes the wedge to broaden as it erodes, which we indicate by the white dotted wedge. (b) For an obtuse opening angle, the shear stress increases downstream, and the wedge tends to become more narrow at later times. (c) A right-angled wedge produces uniform shear stress, which allows the shape to be maintained during erosion. from the its wedge-like form, which is well as well for the evolution process more slowly function process.

FIG. 6. Erosion of an initially circular body. (a) Interfaces from the simulation at evenly spaced time intervals of 0.06 *tf*, with time indicated by the scale bar at right (color). As it shrinks, the body forms a quasi-triangular shape with a wedge-like front that points into the flow. (b) Shifting the interfaces to have the same leading point and rescaling to have equal area more clearly reveals the shape change. (c) The same rescaling procedure applied to the experimental interfaces shows similar evolution and terminal shape.

Messages

write 2 fluids equations write 1 fluid equations (the best)

Importance of shear stress boundary layer controls the erosion