Numerical simulation of the improved dynamics of sedimentary river beds with a stochastic Exner equation

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joint work with
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Outline

Context & Motivations

Saint-Venant–Exner equations

Stochastic case

Context
Organisation

Thesis co-supervised by:

- Université d'Orléans: Stéphane Cordier,
- Université Paris 13: Emmanuel Audusse,
- LHSV, LNHE, EDF R&D: Sébastien Boyaval, Nicole Goutal, Magali Jodeau,

and co-financed by:

- AMIES,
- EDF R&D.

Partnership: Team ANGE

- CEREMA,
- Inria Rocquencourt,
- Université Pierre et Marie Curie: LJLL.

Motivations

Framework

Sediments transport is responsible of modification of river beds.
2 processes of sediments transport:

- by suspension: particles can be found on the whole vertical water depth and rarely be in contact with the bed,
- by bedload: particles are moving near the bed by saltation and rolling.

In the following, we only focuse on the **bedload transport**.
In the literature, most of industrial codes use the **Saint-Venant–Exner model**.

Motivations
Stochastic viewpoint

Work initiated during the **CEMRACS 2013**.

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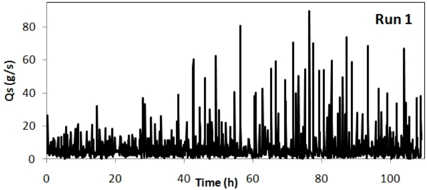
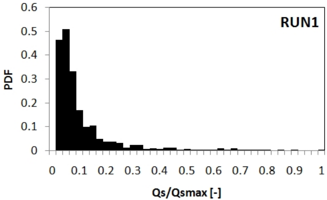
The Exner equation is based in the **conservation of the solid mass** and needs the **definition of a solid flux**. It appears practical but still very coarse.
At the grain scale to the laboratory one, physical experiments reveal **fluctuations of the solid flux** (Recking et al.).



(e) Time variation of solid discharge.



(f) Distribution of solid discharge.

Motivations
Stochastic viewpoint

This problem has been investigated by physical theory and **various stochastic Exner equations** have been proposed (Jerolmack & Mohrig 2005, Ancey 2010 & 2014, Furbish et al. 2012).

We propose a **numerical study of a possible stochastic Exner equation**.

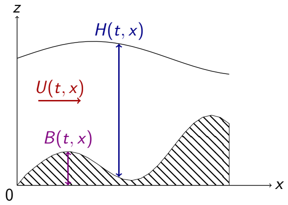
Saint-Venant–Exner equations
The model

Coupled model:

$$\begin{cases} \partial_t H + \partial_x (Q) = 0, & \text{(1a)} \\ \partial_t Q + \partial_x \left(\frac{Q^2}{H} + \frac{g H^2}{2} \right) = -g H \partial_x B - \frac{\tau}{\rho}. & \text{(1b)} \\ \partial_t B + \partial_x Q_s = 0, & \text{(1c)} \end{cases}$$

between:

- the Saint-Venant equations (aka shallow-water equations): (1a)–(1b)



$H(t,x)$: water height,
$Q(t,x) = HU$: discharge,
$B(t,x)$: bottom topography,
with $x \in \Omega \subseteq \mathbb{R}$, $t \geqslant 0$.

Saint-Venant–Exner equations
The model

τ is defined by the **Manning formula**,

$$\tau = \rho g H \frac{Q|Q|}{H^2 K_s^2 R_h^{4/3}}, \quad (2)$$

where, in the particular case of a rectangular channel with width l, the hydraulic radius R_h reads

$$R_h = \frac{lH}{l + 2H}.$$

Saint-Venant–Exner equations
The model

- the Exner equation (1c)

where $Q_s(t,x)$ is the solid transport flux defined by

$$Q_s = \sqrt{\frac{g(\rho_s - \rho)d^3}{\rho}} Q_s^\star\left(\tau^\star;\tau_c^\star\right)\frac{\tau^\star}{|\tau^\star|} \qquad (3)$$

and the **Meyer-Peter-Müller formula**,

$$Q_s^\star = A\left(|\tau^\star| - \tau_c^\star\right)^{3/2} \qquad (4)$$

with
$$\begin{cases} A & \text{the characteristic length of a grain jump,} \\ \rho_s,\ \rho & \text{resp. the mass densities of the solid and fluid phases,} \\ g & \text{the gravitational acceleration,} \\ \tau^\star & \text{the shear stress (aka Shields parameter),} \\ \tau_c^\star & \text{the critical value for the initiation of motion,} \\ d & \text{the grain diameter.} \end{cases}$$

Stochastic Exner model
Ref. Ancey, 2010

Initiation of sediment transport occurs when $\tau > \tau_c$.

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$$V = \sqrt{\frac{g(\rho_s - \rho)d}{\rho}}a(\sqrt{\tau^\star} - \sqrt{\tau_c^\star})$$

where V has an exponential distribution and under the local equilibrium $\tilde{n} \propto (\tau^\star - \tau_c^\star)$, we retrieve the **Meyer-Peter-Müller formula**

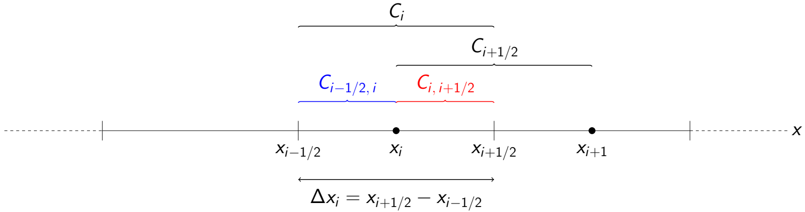
$$Q_s^\star = A\left(|\tau^\star| - \tau_c^\star\right)^{\frac{3}{2}}$$

with A a stochastic coefficient following an **exponential distribution**; $A(t,x)$ **independent in time and space**.

Discretization of Saint-Venant–Exner model

Model defined on the torus: **Periodic boundary conditions**.

Space discretization, $2 \times N_x$ cells denoted $C_{i,\,i+1/2}$, $i = 0, \ldots, N_x - 1$,



- $W_i^n = (H_i^n, Q_i^n)$ defined on C_i,
- $B_{i+1/2}^n$ defined on $C_{i+1/2}$.

Discretization of Saint-Venant–Exner model

1. **Finite Volume method** for the fluid quantities

$$W_i^{n+1} = W_i^n - \frac{\Delta t^n}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right) + \frac{\Delta t^n}{\Delta x} S(W_i^n, B_{i-1/2}^n, B_{i+1/2}^n), \quad (5)$$

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where $F_{i+1/2}$ is the numerical flux defined by the **Rusanov formula**,

$$F_{i+1/2} = F(W_i, W_{i+1}) = \frac{F(W_i) + F(W_{i+1})}{2} - c\frac{W_{i+1} - W_i}{2}, \tag{6}$$

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with $F(W) = (Q, \frac{Q^2}{H} + \frac{gH^2}{2})^T$,

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$c = \max\left(|U_i| + \sqrt{gH_i}, |U_{i+1}| + \sqrt{gH_{i+1}}\right)$ and $S(W_i^n, W_{i+1}^n)$ is a discrete source term written as

$$S(W_i^n, B_{i-1/2}^n, B_{i+1/2}^n) = \begin{pmatrix} 0 \\ gH_i^n(B_{i+1/2}^n - B_{i-1/2}^n) \end{pmatrix}. \tag{7}$$

Discretization of Saint-Venant–Exner model

2. Bottom topography appoximated by a **centered finite difference formula**,

$$B_{i+1/2}^{n+1} = B_{i+1/2}^{n} - \frac{\Delta t^n}{\Delta x}\left(Q_s(H_{i+1}^n, U_{i+1}^n) - Q_s(H_i^n, U_i^n)\right), \quad (8)$$

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Discrete stochastic solid flux given by the stochastic **Meyer-Peter-Müller formula**

$$(Q_s)_i^n = \sqrt{\frac{g(\rho_s - \rho)d^3}{\rho}} A_i^n \left(|(\tau^\star)_i^n| - \tau_c^\star\right)^{\frac{3}{2}} sg((\tau^\star)_i^n)$$

where A_i^n are independent identically distributed random variables with exponential distribution.

Stochastic Saint-Venant–Exner model
Monte-Carlo simulations

Description of the deterministic case

- Stationary uniform flow in torrential regime,
- Sloped bottom topography.

Number of realizations $M = 1000$.

T_{fin} **sufficiently large** such that the empirical variance of all the quantities of interest seems close to long-time stationary values.

Monte-Carlo simulations for the SVE model

Numerical results

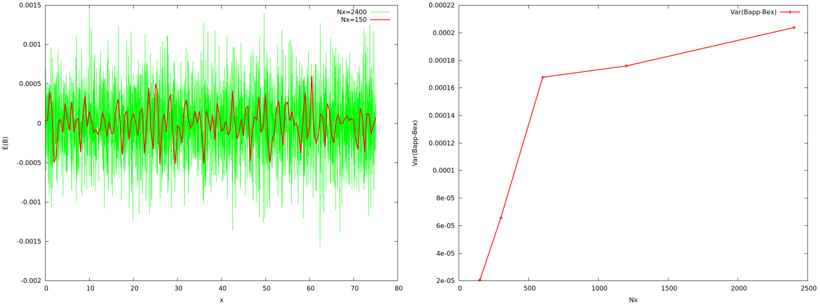

Figure: Empirical mean of the topography deviation for two meshes (left) and empirical variance of the topography deviation for different meshes (right)

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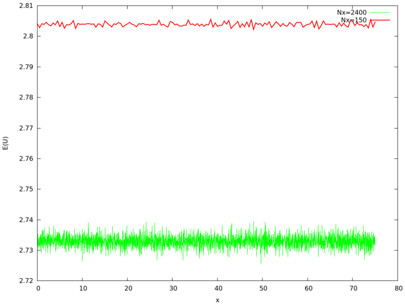

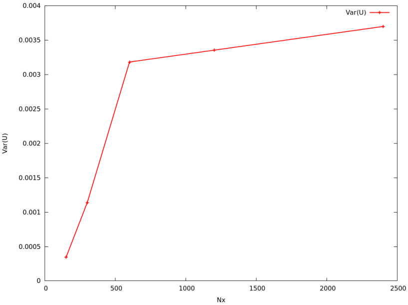

Figure: Empirical mean of the velocity for two meshes (left) and empirical variance of the velocity for different meshes (right)

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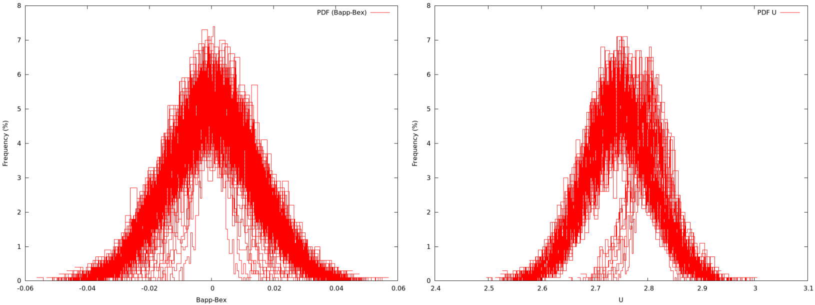

Figure: PDFs for the bottom topography deviation (left) the velocity (right) for the finest mesh

Stochastic Saint-Venant system

Definition

In order to understand the dissipative phenomenon, we propose to return to a simpler problem: **Saint-Venant equations associated with a perturbed topography**.

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Expression of the perturbed topography

$$\begin{cases} B^n_{i+1/2} = B^0_{i+1/2} + \tilde{B}_{i+1/2}, \\ \tilde{B}_{i+1/2} = \alpha\sqrt{\Delta x}\displaystyle\sum_{k=1}^{N/2}\frac{1}{k}\left(a_k \cos\left(2k\pi\frac{i+1/2}{N}\right) + b_k \sin\left(2k\pi\frac{i+1/2}{N}\right)\right), \end{cases}$$

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where $B^0_{i+1/2}$ corresponds to the non-perturbed initial bottom topography,

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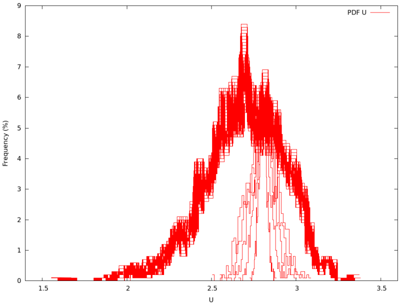
where $B^0_{i+1/2}$ corresponds to the non-perturbed initial bottom topography, a_k and b_k are values obtained with a normal law $\mathcal{N}(0,1)$, and α is an imposed amplitude.

Stochastic Saint-Venant system

Numerical results



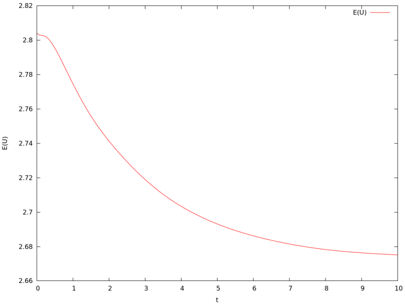


Figure: PDF for the velocity (left) and empirical mean of the velocity as a function of time (right) with $N_x = 150$

Stochastic Saint-Venant system

Calibration with the Strickler coefficient K_s

The Strickler coefficient is calibrated so as to enforce the equilibrium.

Stochastic Saint-Venant system

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Idea

The new coefficient is computed at each time step as a deterministic function of moments of H and Q.
The expression is established by imposing that no energy is dissipated by the model,

$$\mathbb{E}\left[\sum_{i=0}^{N_x-1} Q_i^{n+1}\right] = \mathbb{E}\left[\sum_{i=0}^{N_x-1} Q_i^{n}\right]$$

Stochastic Saint-Venant system

Calibration with the Strickler coefficient K_s

New Strickler coefficient

$$(K_s)^n = \left(- \frac{\mathbb{E}\left[\sum_{i=0}^{N_x-1} \frac{Q_i^{\omega,n} |Q_i^{\omega,n}|}{H_i^{\omega,n} (R_{h,i}^{\omega,n})^{4/3}} \right]}{(-\partial_x B^0) \times \mathbb{E}\left[\sum_{i=0}^{N_x-1} H_i^{\omega,n} \right] + \frac{1}{\Delta x} \mathbb{E}\left[\sum_{i=0}^{N_x-1} H_i^{\omega,n} (\tilde{B}_{i+1/2}^{\omega} - \tilde{B}_{i-1/2}^{\omega}) \right]} \right)^{1/2} \tag{9}$$

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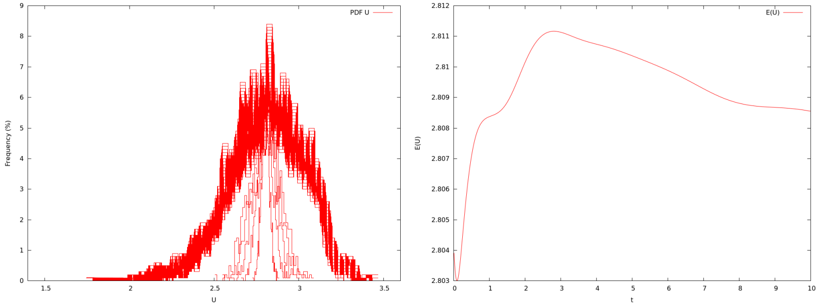

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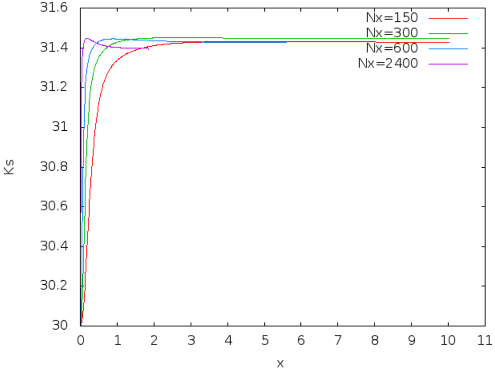

Figure: Strickler coefficient K_s as a function of time for different meshes

Perspectives

- Modelling the noise introduced in Saint-Venant–Exner so as to compensate for the dissipation introduced: scaling of the injected noise with respect to time and space?
- Ability to maintain the deterministic equilibrium in the mean when the bottom topography is perturbed in time in a stochastic Saint-Venant–Exner.
- Robustness of the new model concerning the convergence to a continuous time-space model.
- Comparison of the physical information supported by the stochastic variable A (characteristic size of ripples. . .) with other ones ($\tau^\star$).

Thank you for your attention!