Numerical schemes for viscoplastic avalanches

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Outline

Introduction : Wet snow avalanche, Oisans, 2013

Courtesy P. Etard.

Introduction - Thin layers of viscoplastic fluids

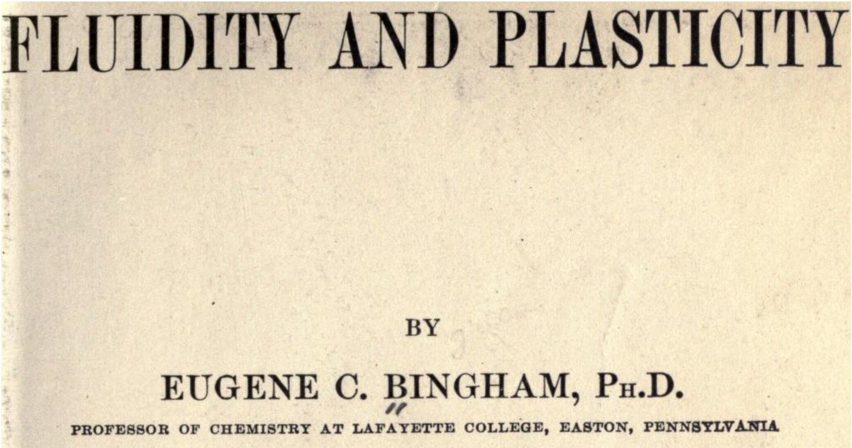
Present objectives

1. **Simulate viscoplastic flows** : with Bingham constitutive law
2. **Thin layers on inclined planes** : lubrication or shallow-water models to reduce computational cost
3. **Schemes able to catch stationary states**

a blend of variational inequalities, finite-volumes schemes and well-balanced philosophy

Plasticity - The origin

1916 ; 1922 →



FLUIDITY AND PLASTICITY

BY

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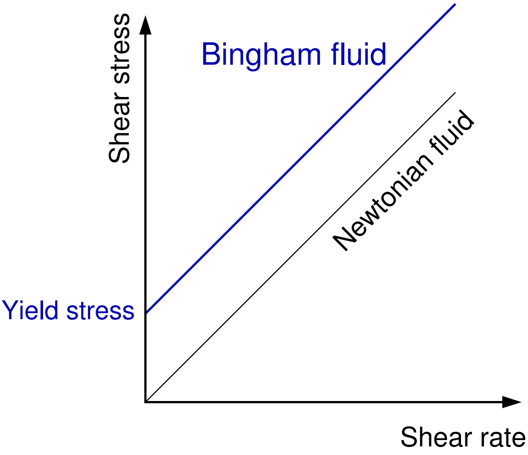
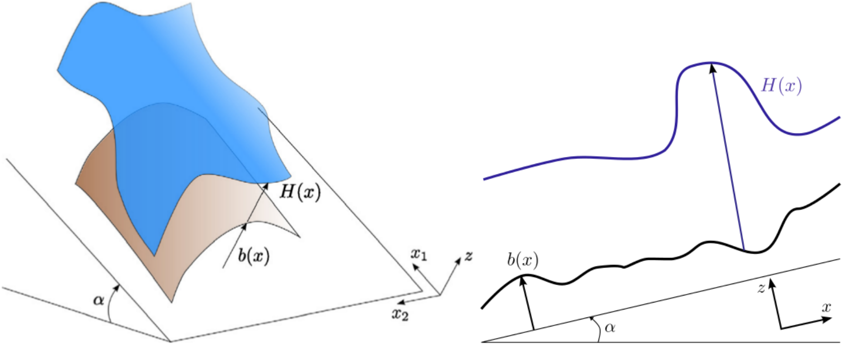
Plasticity

Modern formalism $\rightarrow$



Outline

Asymptotic model - Domain description



Asymptotic model - Domain description

We consider a **general bottom** for the solid boundary.

For this, let $\Omega \subset \mathbb{R}^2$ be a fixed bounded domain and

$$\mathcal{D}(t) = \{(x,z) \in \Omega \times \mathbb{R} \;/\; b(x) < z < b(x) + h(t,x)\},$$

where $h(t,x)$ **is the thickness of the fluid** and $x = (x_1, x_2)$.

$$\Gamma_s(t) := \{(x,z)\;;\; x \in \Omega,\; z = h(t,x)\}, \quad \Gamma_b(t) := \partial\mathcal{D}(t) \setminus \Gamma_s(t)$$

the **free** and **bottom** surfaces. $\boldsymbol{v} := (v_1, v_2)$, the **horizontal** component of the velocity field and w, the **vertical** one, i.e. $\boldsymbol{u} = (\boldsymbol{v}, w)$.

Asymptotic model - Derivation at a glance

$$
\begin{aligned}
&\forall \boldsymbol{\Psi},\ \int_\Omega H\overline{\rho_0}\Big(\mathrm{St}\partial_t \boldsymbol{V}_0\cdot(\boldsymbol{\Psi}-\boldsymbol{V}_0)+\boldsymbol{V}_0\cdot\nabla_x\boldsymbol{V}_0(\boldsymbol{\Psi}-\boldsymbol{V}_0)\Big)dX\\
&+\int_\Omega \beta\boldsymbol{V}_0\cdot(\boldsymbol{\Psi}-\boldsymbol{V}_0)dX\\
&+\int_\Omega \frac{2}{\mathrm{Re}}H\eta D(\boldsymbol{V}_0):D(\boldsymbol{\Psi}-\boldsymbol{V}_0)dX\\
&+\int_\Omega \frac{2}{\mathrm{Re}}H\eta\,\mathrm{div}_x\boldsymbol{V}_0(\mathrm{div}_x\boldsymbol{\Psi}-\mathrm{div}_x\,\boldsymbol{V}_0)dX\\
&+\int_\Omega \tau_y \mathrm{B}H\Big(\sqrt{|D(\boldsymbol{\Psi})|^2+(\mathrm{div}_x\boldsymbol{\Psi})^2}-\sqrt{|D(\boldsymbol{V}_0)|^2+(\mathrm{div}_x\,\boldsymbol{V}_0)^2}\Big)dX\\
&\geq \frac{1}{\mathrm{Fr}^2}\int_\Omega H\overline{\rho_0\mathbf{F}_\Omega}\cdot(\boldsymbol{\Psi}-\boldsymbol{V}_0)dX-\frac{1}{\mathrm{Fr}^2}\int_\Omega (H)^2\overline{Z\rho_0 f_z}(\mathrm{div}_x\boldsymbol{\Psi}-\mathrm{div}_x\boldsymbol{V}_0)dX
\end{aligned}
\tag{1}
$$

Rk1 : $\tau_y = 0$: classical 2D viscous SW, cf. Gerbeau-Perthame
Rk2 : for more details on model derivation $\rightarrow$ Bresch et al. Advances in Math. Fluid Mech. pp 57-89. 2010

Asymptotic model - 1D version

$(x,t) \in [0,L] \times [0,T]$. $H = H(x,t)$, etc. $+ \overline{\rho_0} = cte$
External forces : $f_x = -g \sin\alpha, \quad f_z = -g\cos\alpha.$

$$\frac{\partial H}{\partial t} + \frac{\partial (HV)}{\partial x} = 0, \tag{2}$$

$$\begin{aligned}
&\int_0^L H\Big(\partial_t V(\Psi - V) + \frac{1}{2}\partial_x(V^2)(\Psi - V)\Big)dx \\
&+ \int_0^L \beta V(\Psi - V)dx + \int_0^L 4\eta H\partial_x(V)\partial_x(\Psi - V)dx \\
&+ \int_0^L \tau_y\sqrt{2}H\Big(|\partial_x(\Psi)| - |\partial_x(V)|\Big)dx \\
&\geq \int_0^L H(f_\Omega + f_z\partial_x b)(\Psi - V)dx + \int_0^L \frac{H^2}{2} f_z(\partial_x\Psi - \partial_x V)dx, \quad \forall\Psi
\end{aligned} \tag{3}$$

Outline

Semi-discretization in time

$$\frac{H^{n+1} - H^n}{\Delta t} + \frac{\partial (H^n V^n)}{\partial x} = 0, \tag{4}$$

$$\int_0^L H^n \left(\frac{V^{n+1} - V^n}{\Delta t} (\Psi - V^{n+1}) + \frac{1}{2} \partial_x ((V^n)^2)(\Psi - V^{n+1}) \right) dx$$
$$+ \int_0^L \beta V^{n+1} (\Psi - V^{n+1}) dx + \int_0^L \tau_y \sqrt{2} H^n \Big(|\partial_x \Psi| - |\partial_x V^{n+1}| \Big) dx$$
$$+ \int_0^L 4\eta H^n \partial_x (V^{n+1}) \partial_x (\Psi - V^{n+1}) dx \geq$$
$$\int_0^L H^n \Big(f_\Omega + f_z \, \partial_x b \Big) (\Psi - V^{n+1}) dx - \int_0^L \frac{(H^n)^2}{2} f_z (\partial_x \Psi - \partial_x V^{n+1})$$

Observe : problems on H^{n+1} and V^{n+1} are decoupled

Velocity problem - Augmented Lagrangian

Following Glowinski et al. ('83, '07) $\Rightarrow$ minimization problem

Augmented Lagrangian func. s.t. its saddle point is the solution

Uzawa like algorithm to find this saddle point :

- Solve $\partial_V \mathcal{L}_r(V, q, \mu) = 0$ for V (linear problem)
- $\mathcal{L}_r(V, q, \mu)$ non differentiable in q
  but q can be solved explicitly
- Update the Lagrange multiplier μ and loop

Convergence to the unique solution $\Rightarrow V^{n+1}$

Velocity problem - Augmented Lagrangian

Compute q_i^{k+1} locally (at $\{x_i\}_i$) :

$$q^{k+1} = \begin{cases} 0 \quad \text{if } |\mu^k + r\partial_x(V^k)| < \tau_y, \\ \frac{1}{r}\left((\mu^k + r\partial_x(V^k)) - \tau_y \sqrt{2}\,\mathrm{sgn}(\mu^k + r\partial_x(V^k))\right) \text{otherwise} \end{cases} \tag{6}$$

Solve for V^{k+1} the linear system :

$$H^n\left(\frac{V^{k+1} - V^n}{\Delta t}\right) + \beta V^{k+1} - \partial_x\left(4\eta H^n \partial_x(V^{k+1})\right) - \partial_x\left(rH^n\partial_x(V^{k+1})\right)$$
$$= (f_\Omega + f_z\,\partial_x b)\,H^n + \partial_x\left(f_z\frac{(H^n)^2}{2}\right) - \frac{H^n}{2}\partial_x((V^n)^2) + \partial_x(H^n(\mu^k - rq^{k+1})) \tag{7}$$

Update Lagrange multiplier :

$$\mu^{k+1} = \mu^k + r\left(\partial_x V^{k+1} - q^{k+1}\right) \tag{8}$$

Known facts I

For L.A. methods, the parameter r in

$$\mu^{k+1} = \mu^k + r\left(\partial_x V^{k+1} - q^{k+1}\right)$$

is known to influence the speed of convergence of the algorithm

i.e. the number of iterations to reach V^{n+1}

Known facts II

There exists an optimal r in practice :

$r \to +\infty$ is prevented by the fact that

it also deteriorates condition number of the linear pb

For L.A. methods, except in very simple cases,

no way to derive the optimal r.

An alternative

Bermudez - Moreno method : Comput. Math. Appl., 7(1) :43-58, 1981.
"Duality methods for solving variational inequalities."

General structure close to L.A. :

$$\begin{cases} A(V^k) + \omega B^*(B(V^k)) + B^*(\theta^k) = L, \\ \theta^{k+1} = G_\lambda^\omega(B(V^k) + \lambda\theta^k). \end{cases} \tag{9}$$

where G_λ^ω is the Yosida regularization of the operator accounting for the non differentiable term "$|\partial_x(V^k)|$".

In short, for a given problem, one can use a general way to derive the optimal ω, via eigenvalue problems.

Ex : "exact" in 1D ; "numerical" in 2D for Bingham

Velocity problem - Bermudez-Moreno

- Find $V^{k+1} \in \mathcal{V}$ solution of the following linear problem :

$$\left(\frac{H^n}{\Delta t} + \beta\right) V^{k+1} - \partial_x((4\eta H^n + \omega)\, \partial_x V^{k+1}) - \partial_x(\omega \partial_x V^{k+1})$$
$$= \quad \frac{H^n}{\Delta t} V^n - \frac{H^n}{2} \partial_x((V^n)^2) + \frac{1}{2} \partial_x((H^n)^2 f_z) + H^n(f_\Omega + f_z \partial_x b) + \partial_x \theta^k.$$

- Update the so-called BM multiplier θ^{k+1} via $\xi^{k+1} = \partial_x V^{k+1} + \lambda \theta^k$ and

$$\theta^{k+1} = \begin{cases} \frac{-\omega\, \xi^{k+1} + \tau_y \sqrt{2} H^n(x)}{1 - \lambda\, \omega} & \text{if } \xi^{k+1} > \lambda \tau_y \sqrt{2} H^n(x), \\ \frac{\xi^{k+1}}{\lambda} & \text{if } \xi^{k+1} \in [-\lambda \tau_y \sqrt{2} H^n(x), \lambda\, \tau_y \sqrt{2} H^n(x)], \\ \frac{-\omega\, \xi^{k+1} - \tau_y \sqrt{2} H^n(x)}{1 - \lambda\, \omega} & \text{if } \xi^{k+1} < -\lambda \tau_y \sqrt{2} H^n(x). \end{cases} \tag{10}$$

Note that this computation is again local in space.

$$\omega_{\text{opt}}(H^n_{\max}) = \left(\frac{H^n_{\max}}{\Delta t} + \beta\right) \frac{L^2}{N \pi^2} + 4\eta H^n_{\max}. \tag{11}$$

Height problem and Spatial discretization

Looking at the global problem ... $(P)^{n,k}$:

$$\begin{cases} \frac{H^{k+1}-H^n}{\Delta t} + \partial_x(H^n V^n) = 0, \\ H^n \left(\frac{V^{k+1}-V^n}{\Delta t}\right) + \beta V^{k+1} - \partial_x \left(4\eta H^n \partial_x(V^{k+1})\right) - \partial_x \left(rH^n \partial_x(V^{k+1})\right) \\ = (f_\Omega + f_z\, \partial_x b)\, H^n + \partial_x \left(\frac{(H^n)^2 f_z}{2}\right) - \frac{H^n}{2}\partial_x((V^n)^2) + \partial_x(H^n(\mu^k - rq^{k+1})). \end{cases} \tag{12}$$

... & invoking the "SWE structure"

with various source terms, including the duality terms

$\Rightarrow$ we choose **finite-volume** for spatial discretization of $(P)^{n,k}$

"Unified" viscoplast. Well-Balanced finite volume

- System form :

$$D(W^n)\left(\frac{W^{k+1}-W^n}{\Delta t}+\partial_x F(W^n)\right)-\partial_x((4\eta H^n+\delta^n)\underline{\mathcal{I}}\,\partial_x W^{k+1})$$
$$=-\beta\underline{\mathcal{I}}\,W^{k+1}+S(W^n)\partial_x\overline{\sigma}^k,$$

- Flux approximation :

$$\phi(W_i^n, W_{i+1}^n, \{\zeta^k_{j+1/2}\}_{j=i-1}^{j=i+1}) = \frac{F(W_i^n)+F(W_{i+1}^n)}{2} - \frac{1}{2}Q^n_{i+1/2}(W^n_{i+1}-W^n_i+\mathcal{G}(\{\zeta^k_{j+1/2}\}_{j=i-1}^{j=i+1}))$$

- Numerical viscosity matrix, $Q^n_{i+1/2}$, various possibilities :
  (modified) Lax-Friedrichs, Rusanov [diagonal],
  Roe, HLL, Lax-Wendroff, Force, Gforce [not diagonal].

$\leftarrow$ source terms treatment of *Chacón et al.* SIAM JSC 2007

Distributing the system "on speed and height"

Linear (sub)problem on V

$$\boldsymbol{A}^n V^{k+1} = \boldsymbol{b}^{n,k}, \tag{13}$$

$$\boldsymbol{b}_i^{n,k} = \boldsymbol{b}_i^{n,(1)} + \boldsymbol{b}_i^{n,k,(2)} + \boldsymbol{b}_i^{n,k,(3)},$$

where

$$\begin{aligned}
\boldsymbol{b}_i^{n,(1)} &= H_i^n \left(f_\Omega + f_z \frac{b_{i+1} - b_{i-1}}{2\,\Delta x} \right), \\
\boldsymbol{b}_i^{n,k,(2)} &= \frac{\zeta^k_{i+1/2} - \zeta^k_{i-1/2}}{\Delta x}, \\
\boldsymbol{b}_i^{n,k,(3)} &= H_i^n \frac{[\phi(W^n_{i-1}, W^n_i, \{\zeta^k_{j+1/2}\}_{j=i-1}^{j=i+1})]_2 - [\phi(W^n_i, W^n_{i+1}, \{\zeta^k_{j+1/2}\}_{j=i-1}^{j=i+1})]_2}{\Delta x}
\end{aligned}$$

Rk : If $Q^n_{i+1/2}$ diagonal, $\boldsymbol{b}_i^{n,k,(3)} \rightsquigarrow \boldsymbol{b}_i^{n,(3)}$. Reduce comput. cost

Distributing the system “on speed and height”

Sub-Problem on H

$$D(W_i^n)\left(\frac{W_i^{k+1}-W_i^n}{\Delta t}+\frac{\phi(W_i^n,W_{i+1}^n,\{\zeta_{j+1/2}^k\}_{j=i-1}^{j=i+1})-\phi(W_{i-1}^n,W_i^n,\{\zeta_{j+1/2}^k\}_{j=i-2}^{j=i})}{\Delta x}\right)$$
$$-\frac{1}{\Delta x^2}\left((4\eta H_{i+1/2}^n+\delta_{i+1/2}^n)\underline{\mathcal{I}}\,(W_{i+1}^{k+1}-W_i^{k+1})-(4\eta H_{i-1/2}^n+\delta_{i-1/2}^n)\underline{\mathcal{I}}\,(W_i^{k+1}-W_{i-1}^{k+1})\right)$$
$$=-\beta\underline{\mathcal{I}}\,W_i^{k+1}+S(W_i^n)\frac{\overline{\sigma}_{i+1/2}^{n,k}-\overline{\sigma}_{i-1/2}^{n,k}}{\Delta x}. \quad (14)$$

where

$$H_{i+1/2}^n=\frac{H_i^n+H_{i+1}^n}{2},\quad \overline{\sigma}_{i+1/2}^{n,k}=\begin{pmatrix} f_\Omega\, x_{i+1/2}+f_z\,\frac{b_i+b_{i+1}}{2} \\ \zeta_{i+1/2}^k \end{pmatrix},$$

$$\mathcal{G}(\{\zeta_{j+1/2}^k\}_{j=i-1}^{j=i+1})=\frac{1}{f_z}\begin{pmatrix} f_\Omega\,\Delta x+f_z\,(b_{i+1}-b_i)+\frac{\Delta(\zeta+\delta^n\,\partial_x V)_{i+1/2}^k}{H_{i+1/2}} \\ 0 \end{pmatrix},$$

Compute $H^{n+1}=H^{k+1}$ with the *first* component of (14) and the *most recent* "duality multiplier", i.e, ζ^{n+1}

Well-Balancing and Wet/Dry treatment

To design $\Delta(\zeta + \delta^n \partial_x V)^k_{i+1/2}$, use a flux limiter in conjunction with a convex combination of a 2nd order and a 1st order approximation of $\partial_x(\zeta + \delta^n \partial_x V)^k_{i+1/2}$

Wet/Dry correction : check whether rigid or fluid

- no numerical diffusion in H discretiz.
- and local equilibrium of pressure

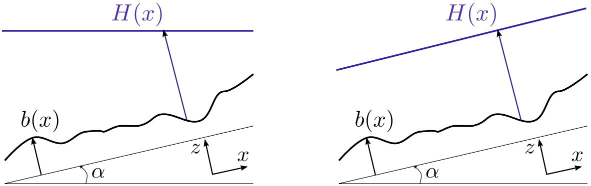
Properties :

- recover standard w/d treatment [WD] when fluid
- and natural resting state when rigid

[WD] Castro et al. *Mathematical and Computer Modelling* **42**(3-4), 2005.

Well-Balanced property of the coupled scheme

Two stationary solutions : $V = 0$ and



If $\left| f_\Omega \Delta x \left(\sum_{j=1}^{i} H_j^0 - \sum_{j=1}^{[N/2]} H_j^0 \right) \right| \le \tau_y \sqrt{2} H^0_{i+1/2} \forall\, i$, then ↑ is stat. sol. **(25)**

Theorem

Let $(H = H(x);\ V \equiv 0)$ *be a stationary solution of (3), and assume that the proposed numerical scheme uses the following initialization for* ζ *:*

$$\zeta^1_{i+1/2} = -\Delta x \left(\sum_{j=1}^{i} H_j^0 - \sum_{j=1}^{[N/2]} H_j^0 \right) \left(f_\Omega + f_z \frac{H^0_{i+1} + b_{i+1} - (H^0_i + b_i)}{\Delta x} \right),$$

then, the scheme exactly preserves both stationary solutions : (i) horizontal free surface and (ii) flat height, verifying (25), over a bottom b.

Outline

The duct flow case

An analytical solution for Poiseuille-Bingham - $H = 1$ and

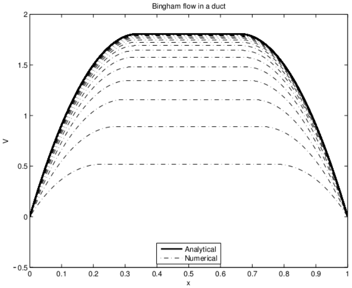
$$\forall \Psi, \int_0^L \partial_t V(\Psi - V) + 4\eta \partial_x(V)\partial_x(\Psi - V) + \int_0^L \tau_y \sqrt{2}\left(|\partial_x \Psi| - |\partial_x V|\right) dx \geq \int_0^L f(\Psi - V) dx. \quad (15)$$

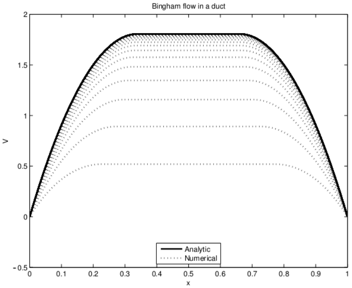
where f is the pressure gradient in the direction of the flow.
Analytical stationary solution :

$$V_{BP}(\chi) = \frac{f}{8\eta} \begin{cases} \left(\frac{L}{2} - \chi_y\right)^2 & \text{if } 0 \leq \chi \leq \chi_y, \\ \left(\frac{L}{2} - \chi_y\right)^2 - (\chi - \chi_y)^2 & \text{if } \chi_y < \chi \leq \frac{L}{2}. \end{cases} \quad (16)$$

where $\xi = |x - \frac{L}{2}|$, $\xi_o = \sqrt{2}\tau_y / f$ and the domain is $x \in [0, L]$.

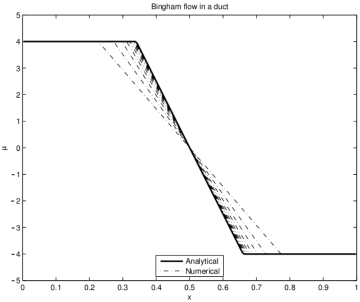
The duct flow case

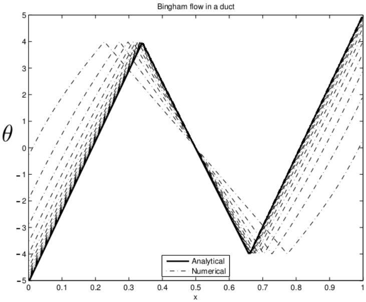




Convergence of order 2 in space for both AL and BM

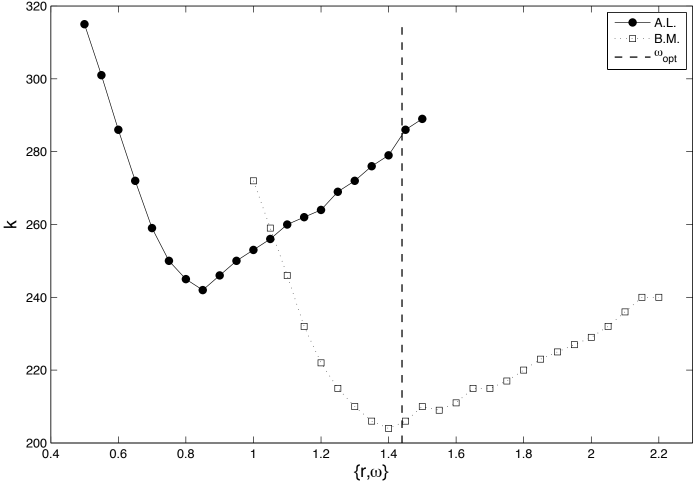
The duct flow case



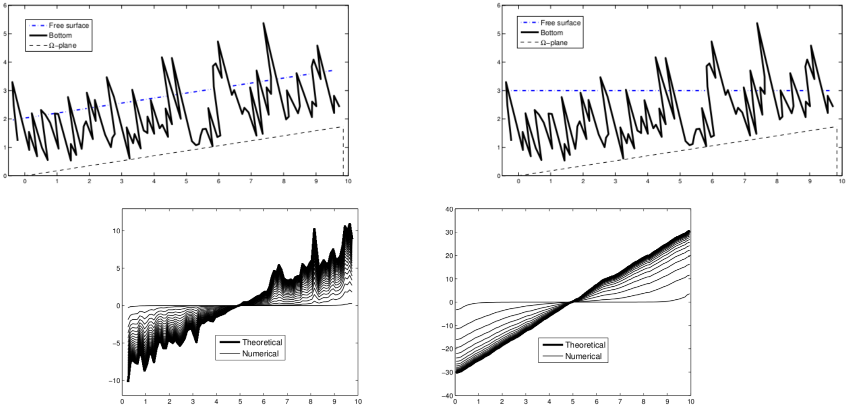


Duality variables : AL and BM

Optimal parameters



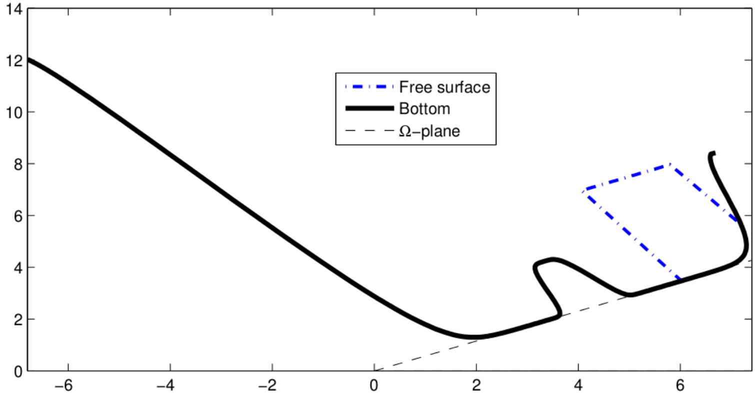
Well-Balanced : 2 stationary solutions



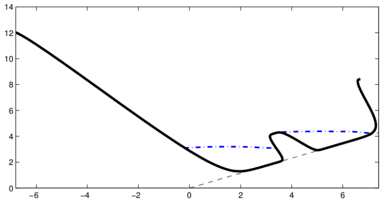
(a) $\mu(x)$

(b) $\theta(x)$

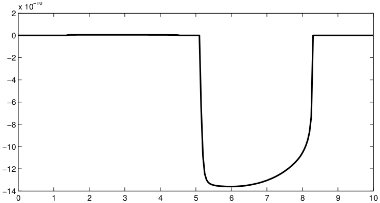
Baby avalanche : test 3. Initial condition



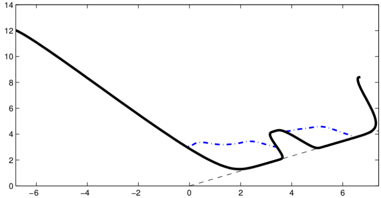
Baby avalanche : test 3. Final time (a)



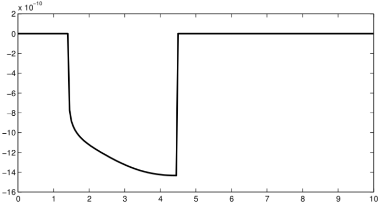
(a) $\tau_y = 1$. Free surface.



(b) $\tau_y = 1$. Velocity.

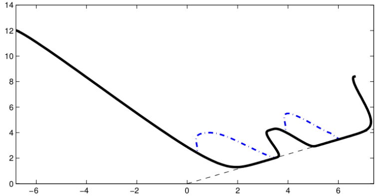


(c) $\tau_y = 4$. Free surface.

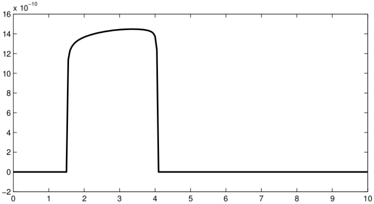


(d) $\tau_y = 4$. Velocity.

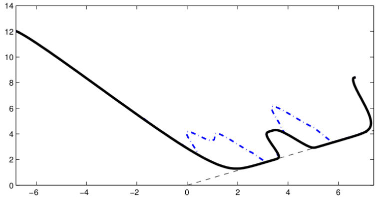
Baby avalanche : test 3. Final time (b)



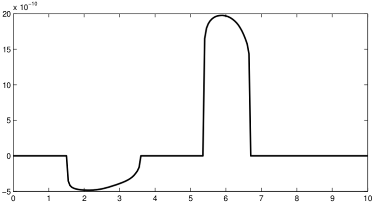
(e) $\tau_y = 8$. Free surface.



(f) $\tau_y = 8$. Velocity.

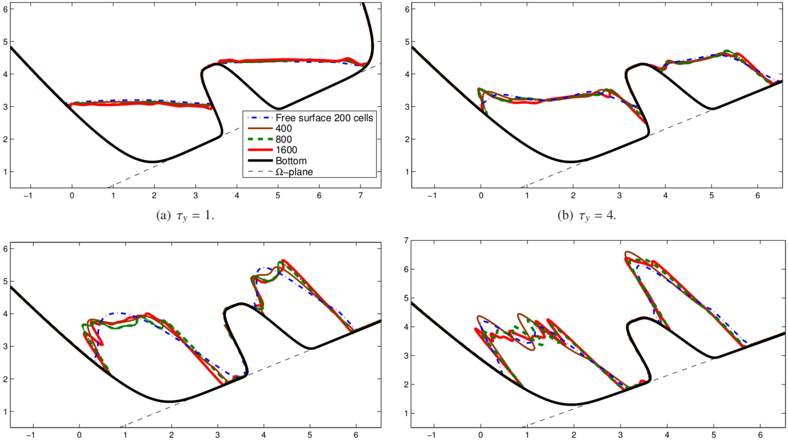


(g) $\tau_y = 12$. Free surface.



(h) $\tau_y = 12$. Velocity.

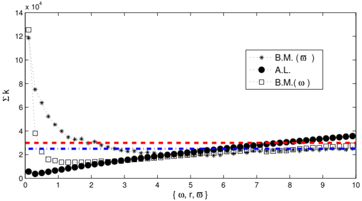
Baby avalanche : test 3. Final time, mesh conv.



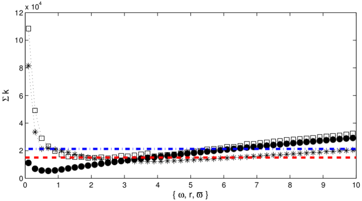
(a) $\tau_y = 1$. (b) $\tau_y = 4$.

(c) $\tau_y = 8$. (d) $\tau_y = 12$.

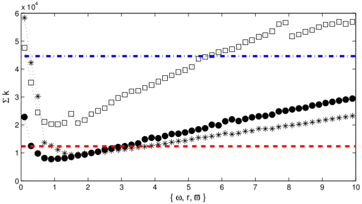
Baby avalanche : test 3. Comput. cost



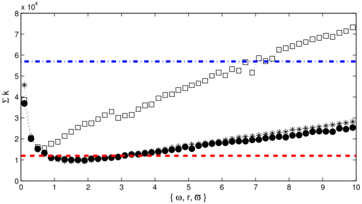
(a) $\tau_y = 1$.



(b) $\tau_y = 4$.



(c) $\tau_y = 8$.



(d) $\tau_y = 12$.

Conclusions

Summary

- derivation of a Shallow-Water Bingham model
  - Re = $\mathcal{O}(1)$ and 1^{st} order slip BC
  - valid for null slope and up to moderate slopes
- design of **Well-Balanced schemes** which allow to catch stationary states by **coupling duality methods** and **Finite-Volume methods**...
- ... taking into acc. wet/dry fronts on general slopes
- BM : a way to determine optimal duality parameters

Perspectives

- extension of such schemes to simulations in 2D space

More details : J. of Comput. Phys. (2014) Vol 64, pp 55-90