Absorbing Boundary Condition, Domain Decomposition and Hydrodynamic Wave Model

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M2N/IMath/Cnam

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le cnam

Introduction

The hydrodynamic wave model (Ph. D. [1]) (without flow):

$$
\begin{cases}
\partial_t^2\eta - c_r^2\partial_x^2\eta + \dfrac{g}{2\varepsilon}\eta + \dfrac{1}{2\varepsilon}\dot{\varphi} = 0 \text{ on } \Gamma_s, c_r = \sqrt{\sigma/(2\varepsilon\rho)},\\
\partial_t^2\varphi - c_f^2\Delta\varphi = 0 \text{ in } \Omega,\\
\dfrac{\partial\varphi}{\partial\nu} = \partial_t\eta \text{ on } \Gamma_s,\\
+ \text{I.C.}
\end{cases}
$$

Tests with the Engquist-Majda boundary condition of order one [2], interesting results for initial conditions in the domain not on the surface.

So we want to test a high-order boundary condition (Higdon ... Hagstrom - Warburton).

[1] ... Neumann-Kelvin ... - Ana. App., Vol. 8, No. 4 (2010) 1–26

[2] mémoire Cnam - D.M. Bissengue, PH. Destuynder, O. Wilk

A history ... (1/3)

- Wave eq. (Engquist-Majda (1977), Higdon (86)),
- dispersive wave eq. (Hagstrom (99), Hagstrom, Mass-or, Givoli (2008)),
- with flow (Bécache, Givoli, Hagstrom (2010)).

High-order boundary condition of order $p+1$:

$$(a_p\partial_t+c\partial_\nu)...(a_1\partial_t+c\partial_\nu)(a_0\partial_t+c\partial_\nu)u=0, a_i \leq 1, i=0,p.$$

Not easy to use.

A history ... (2/3)

Auxiliary variable formulations :

- Givoli, Neta (2003) (unstable for $t \nearrow$) :

$$\begin{aligned}
(a_0\partial_t + c\partial_\nu)u &= \phi_1, \\
(a_1\partial_t + c\partial_\nu)\phi_1 &= \phi_2, \\
&\vdots \\
\phi_{p+1} &= 0.
\end{aligned}$$

- Hagstrom, Warburton (2004) (more stable) :

$$\begin{aligned}
(a_0\partial_t + c\partial_\nu)u &= a_0\partial_t\phi_1, \\
(a_1\partial_t + c\partial_\nu)\phi_1 &= (a_1\partial_t - c\partial_\nu)\phi_2, \\
&\vdots \\
\phi_{p+1} &= 0.
\end{aligned}$$

Equivalent to :

$$(a_0\partial_t + c\partial_\nu)\left[\prod_{j=1}^{p}(a_j\partial_t + c\partial_\nu)^2\right]u = 0$$

A history ... (3/3)

An example with the dispersive problem :

$$\partial_t^2 u - c^2 \partial_x^2 u - c^2 \partial_y^2 u + f^2 u = 0 \text{ in } \mathbb{R} + C.I.$$

Linearity (boundary cond.) + I.C. equal to zero ($a_j = 1$), we get :

$$\partial_t^2 \phi_j - c^2 \partial_x^2 \phi_j - c^2 \partial_y^2 \phi_j + f^2 \phi_j = 0, \forall j = 1, p.$$

Replace "$c^2 \partial_x^2 \phi_j$" (ex. with the Givoli-Neta formulation) :

$$\begin{array}{rccc} c^2 \partial_x^2 \phi_j = & (c\partial_x - \partial_t) & (c\partial_x + \partial_t)\phi_j & + \partial_t^2 \phi_j \\ c^2 \partial_x^2 \phi_j = & (c\partial_x - \partial_t) & \phi_{j+1} & + \partial_t^2 \phi_j \\ \text{et} & (c\partial_x + \partial_t) & \phi_{j+1} & = \phi_{j+2} \end{array}$$

we get just on the boundary ($\forall j = 1, p$, $\phi_0 = u$) :

$$\partial_t^2 \phi_{j-1} + 2\partial_t \phi_j - \phi_{j+1} - c^2 \partial_y^2 \phi_{j-1} + f^2 \phi_{j-1} = 0.$$

Absorbing Boundary Condition and Hydrodynamic Wave Model

A coupled model.

H.W. A.B.C. with the surface model

$$\partial_t^2\eta - c_r^2\partial_x^2\eta + \frac{g}{2\varepsilon}\eta + \frac{1}{2\varepsilon}\dot{\varphi} = 0 \text{ on } \Gamma_s \Longleftrightarrow \mathcal{L}_1(\eta) + \mathcal{L}_2(\varphi) = 0 \text{ on } \Gamma_s,$$

$$\begin{array}{cc}
(a_{r0}\partial_t + c_r\partial_\nu)\eta = a_{r0}\partial_t\phi_1^\eta, & (a_{f0}\partial_t + c_f\partial_\nu)\varphi = a_{f0}\partial_t\phi_1^\varphi, \\
\mathcal{L}_1((a_{r0}\partial_t + c_r\partial_\nu)\eta) = \mathcal{L}_1(a_{r0}\partial_t\phi_1^\eta), & \mathcal{L}_2((a_{f0}\partial_t + c_f\partial_\nu)\varphi) = \mathcal{L}_2(a_{f0}\partial_t\phi_1^\varphi), \\
(a_{r0}\partial_t + c_r\partial_\nu)\mathcal{L}_1(\eta) = a_{r0}\partial_t\mathcal{L}_1(\phi_1^\eta), & (a_{f0}\partial_t + c_f\partial_\nu)\mathcal{L}_2(\varphi) = a_{f0}\partial_t\mathcal{L}_2(\phi_1^\varphi), \\
(\partial_t + C_r\partial_\nu)\mathcal{L}_1(\eta) = \partial_t\mathcal{L}_1(\phi_1^\eta), & (\partial_t + C_f\partial_\nu)\mathcal{L}_2(\varphi) = \partial_t\mathcal{L}_2(\phi_1^\varphi).
\end{array}$$

If $C_r = C_f = C$, we get :

$$(\partial_t + C\partial_\nu)(\mathcal{L}_1(\eta) + \mathcal{L}_2(\varphi)) = \partial_t(\mathcal{L}_1(\phi_1^\eta) + \mathcal{L}_2(\phi_1^\varphi)),$$

so :

$$\partial_t(\mathcal{L}_1(\phi_1^\eta) + \mathcal{L}_2(\phi_1^\varphi)) = 0 \text{ on } \Gamma_s,$$

with I.C. equal to zero, we get :

$$\mathcal{L}_1(\phi_1^\eta) + \mathcal{L}_2(\phi_1^\varphi) = 0 \text{ on } \Gamma_s.$$

And also, we get $\forall\phi_i, i = 1, p$:

$$\mathcal{L}_1(\phi_i^\eta) + \mathcal{L}_2(\phi_i^\varphi) = 0 \text{ on } \Gamma_s.$$

H.W. A.B.C. with the basin model

In the domain, we get (without condition on the coefficients) :

$$\partial_t^2 \phi_i^{\varphi} - c_f^2 \Delta \phi_i^{\varphi} = 0 \text{ in } \Omega, \forall i = 1, p.$$

But for the condition between the surface model and the domain model :

$$\partial_\nu \varphi = \partial_t \eta \text{ on } \Gamma_s,$$

we get (if $C_r = C_f = C$) :

$$\partial_\nu \phi_i^{\varphi} = \partial_t \phi_i^{\eta} \text{ on } \Gamma_s, \forall i = 1, p.$$

Hydrodynamic Wave Model - H.W. A.B.C.

$$
\left\{
\begin{array}{l}
\partial_t^2 \eta - c_r^2 \partial_x^2 \eta + \frac{g}{2\varepsilon}\eta + \frac{1}{2\varepsilon}\dot{\varphi} = 0 \text{ on } \Gamma_s,\, c_r = \sqrt{\sigma/(2\varepsilon\rho)}, \\
\partial_t^2 \varphi - c_f^2 \Delta\varphi = 0 \text{ in } \Omega, \frac{\partial\varphi}{\partial\nu} = \partial_t \eta \text{ on } \Gamma_s, + \text{I.C.} \\
\\
(a_0\partial_t + c_r\partial_x)\eta = a_0 \partial_t \phi_1^\eta \text{ at } P_{out}, \\
(a_0\partial_t + c_f\partial_x)\varphi = a_0 \partial_t \phi_1^\varphi \text{ on } \Gamma_{out}, \\
\\
\forall j = 1, .., J \text{ at } P_{out} : \\
\quad l_{j,j-1}\, \partial_t^2 \phi_{j-1}^\eta + l_{j,j}\, \partial_t^2 \phi_j^\eta + l_{j,j+1}\, \partial_t^2 \phi_{j+1}^\eta \\
\quad + \frac{g}{2\varepsilon}(m_{j,j-1}\, \phi_{j-1}^\eta + m_{j,j}\, \phi_j^\eta + m_{j,j+1}\, \phi_{j+1}^\eta) \\
\quad + \frac{1}{2\varepsilon}(m_{j,j-1}\, \partial_t \phi_{j-1}^\varphi + m_{j,j}\, \partial_t \phi_j^\varphi + m_{j,j+1}\, \partial_t \phi_{j+1}^\varphi) = 0, \\
\\
\forall j = 1, .., J \text{ on } \Gamma_{out} : \\
\quad l_{j,j-1}\, \partial_t^2 \phi_{j-1}^\varphi + l_{j,j}\, \partial_t^2 \phi_j^\varphi + l_{j,j+1}\, \partial_t^2 \phi_{j+1}^\varphi \\
\quad - c_f^2 (m_{j,j-1}\, \partial_y^2 \phi_{j-1}^\varphi + m_{j,j}\, \partial_y^2 \phi_j^\varphi + m_{j,j+1}\, \partial_y^2 \phi_{j+1}^\varphi) = 0, \\
\\
\frac{\partial \phi_j^\varphi}{\partial\nu} = \partial_t \phi_j^\eta \text{ at } P_{out}, \frac{\partial \phi_j^\varphi}{\partial\nu} = 0 \text{ at the other end.}
\end{array}
\right.
$$

Wave speed on the surface : **useful for a (1/2)**

Top $y = 0$, bottom $y = h, h < 0$ [1]:

$$\begin{cases} (i): \quad 2\varepsilon\partial_t^2\eta - \frac{\sigma}{\rho}\partial_x^2\eta + g\eta + \dot{\varphi} = 0 \text{ on } \Gamma_s, (\varepsilon \text{ small}) \\ (ii): \quad \partial_t^2\varphi - c_f^2\Delta\varphi = 0 \text{ in } \Omega, \text{ with } \frac{\partial\varphi}{\partial\nu} = \partial_t\eta \text{ on } \Gamma_s \end{cases}$$

$\partial_t(i)$ et $\partial_t\eta = \partial_\nu\varphi : -\frac{\sigma}{\rho}\partial_x^2\partial_y\varphi + g\partial_y\varphi + \ddot{\varphi} = 0$ on Γ_s,

with $\varphi = A(y)\exp^{i(kx-\omega t)}$ (remark : $C = \omega/k$) :

$$\begin{cases} (iii): \quad (g + \frac{\sigma}{\rho}k^2)\dot{A}(y) - \omega^2 A(y) = 0 \text{ on } \Gamma_s(y=0), \\ (iv): \quad \ddot{A}(y) - l^2A(y) = 0 \text{ dans } \Omega, \text{ with } l = \sqrt{\underbrace{\frac{c_f^2k^2 - \omega^2}{c_f^2}}_{\text{hyp. : } c_fk > \omega}} \end{cases}$$

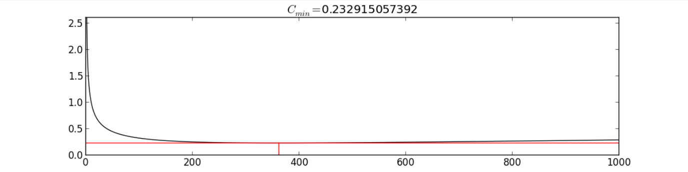
$$\begin{aligned} (iv) \Rightarrow A(y) &= K_1\exp(ly) + K_2\exp(-ly), \\ &= K_1\exp(ly) \text{ small } \partial_\nu|_{y=h} = 0, h \to -\infty \text{ and } |y| \text{small.} \end{aligned}$$

(iii) if $l \simeq k \Rightarrow C = \frac{\omega}{k} = \pm\sqrt{\frac{g}{k} + \frac{\sigma}{\rho}k}$ with $C_{min} = (\frac{4\sigma g}{\rho})^{1/4}$.

[1] J. Fabre

Wave speed on the surface : **useful for** a (2/2)

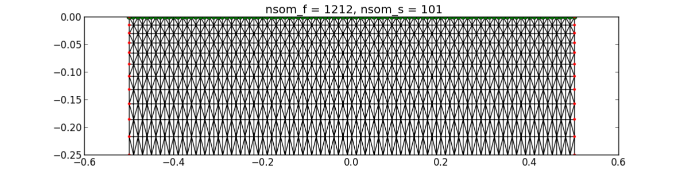
$C(k)$ $(\sigma = 0.075 N.m^{-1}, \rho = 1000 kg.m^{-3}, g = 9.81 m.s^{-2})$:

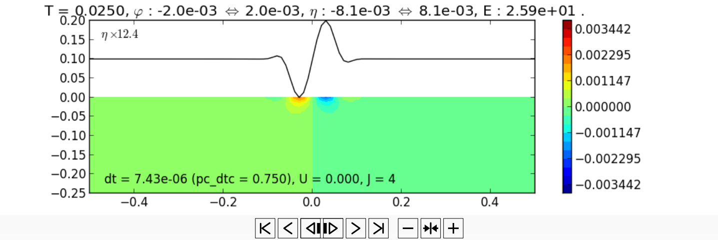


$$\text{If } l = k\sqrt{1 - \frac{\omega^2}{c_f^2 k^2}} \Rightarrow \tilde{C} = C \underbrace{(1 - \frac{\omega^2}{c_f^2 k^2})^{1/4}}_{<1}.$$

So to absorb the waves with a speed $< C_{min}$:
$a_f > c_f / C_{min}$, $a_r = (c_r/c_f) a_f$.

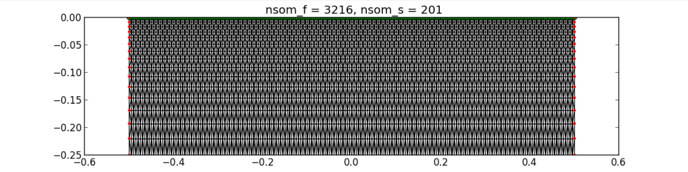
Numerical application : (1/2)

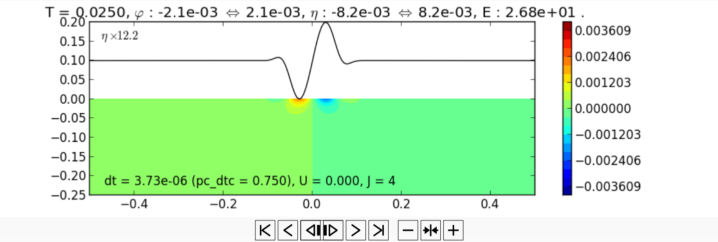




$$c_f = 1000 m.s^{-1}, J = 4, a = 1.3\ c_f / C_{min} - 1.5\ c_f / C_{min}.$$

Numerical application : (2/2)



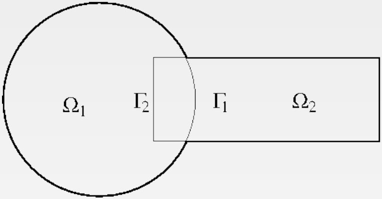


$$c_f = 1000 m.s^{-1}, J = 4, a = 1.8\ c_f / C_{min} - 2.0\ c_f / C_{min}.$$

Absorbing Boundary Condition and Domain Decomposition Method

Bibliography

- Schwarz (1870)

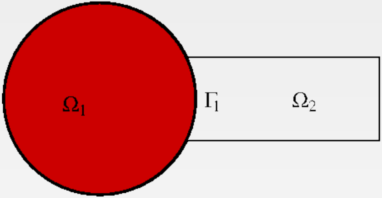


- P.L. Lions (1988) (parallel)

$$
\begin{aligned}
\Delta u_1^n &= 0 && \text{in } \Omega_1 & \Delta u_2^n &= 0 && \text{in } \Omega_2 \\
u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega_1} & u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega_2} \\
u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 & u_2^n &= u_1^{n-1} && \text{on } \Gamma_2
\end{aligned}
$$

Bibliography

- Schwarz (1870)



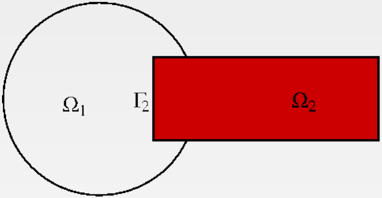
$$
\begin{aligned}
\Delta u_1^1 &= 0 && \text{in } \Omega_1 \\
u_1^1 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\
u_1^1 &= u_2^0 && \text{on } \Gamma_1
\end{aligned}
$$

- P.L. Lions (1988) (parallel)

$$
\begin{aligned}
\Delta u_1^n &= 0 && \text{in } \Omega_1 \qquad & \Delta u_2^n &= 0 && \text{in } \Omega_2 \\
u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \qquad & u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\
u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 \qquad & u_2^n &= u_1^{n-1} && \text{on } \Gamma_2
\end{aligned}
$$

Bibliography

- Schwarz (1870)



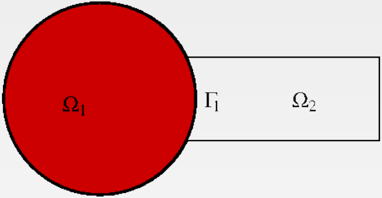
$$\begin{aligned} \Delta u_2^1 &= 0 && \text{in } \Omega_2 \\ u_2^1 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_2^1 &= u_1^1 && \text{on } \Gamma_2 \end{aligned}$$

- P.L. Lions (1988) (parallel)

$$\begin{aligned} \Delta u_1^n &= 0 && \text{in } \Omega_1 \\ u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 \end{aligned} \qquad \begin{aligned} \Delta u_2^n &= 0 && \text{in } \Omega_2 \\ u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_2^n &= u_1^{n-1} && \text{on } \Gamma_2 \end{aligned}$$

Bibliography

- Schwarz (1870)



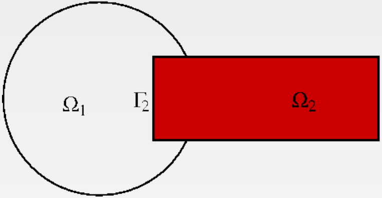
$$
\begin{aligned}
\Delta u_1^2 &= 0 && \text{in } \Omega_1 \\
u_1^2 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\
u_1^2 &= u_2^1 && \text{on } \Gamma_1
\end{aligned}
$$

- P.L. Lions (1988) (parallel)

$$
\begin{aligned}
\Delta u_1^n &= 0 && \text{in } \Omega_1 & \Delta u_2^n &= 0 && \text{in } \Omega_2 \\
u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 & u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\
u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 & u_2^n &= u_1^{n-1} && \text{on } \Gamma_2
\end{aligned}
$$

Bibliography

- Schwarz (1870)



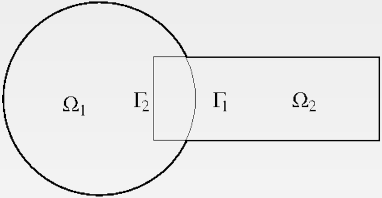
$$
\begin{aligned}
\Delta u_2^2 &= 0 && \text{in } \Omega_2 \\
u_2^2 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\
u_2^2 &= u_1^2 && \text{on } \Gamma_2
\end{aligned}
$$

- P.L. Lions (1988) (parallel)

$$
\begin{aligned}
\Delta u_1^n &= 0 && \text{in } \Omega_1 \\
u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\
u_1^n &= u_2^{n-1} && \text{on } \Gamma_1
\end{aligned}
\qquad
\begin{aligned}
\Delta u_2^n &= 0 && \text{in } \Omega_2 \\
u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\
u_2^n &= u_1^{n-1} && \text{on } \Gamma_2
\end{aligned}
$$

Bibliography

- Schwarz (1870)



$$\begin{aligned} \Delta u_1^n &= 0 && \text{in } \Omega_1 & \Delta u_2^n &= 0 && \text{in } \Omega_2 \\ u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 & u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 & u_2^n &= u_1^n && \text{on } \Gamma_2 \end{aligned}$$

- P.L. Lions (1988) (parallel)

$$\begin{aligned} \Delta u_1^n &= 0 && \text{in } \Omega_1 & \Delta u_2^n &= 0 && \text{in } \Omega_2 \\ u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 & u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 & u_2^n &= u_1^{n-1} && \text{on } \Gamma_2 \end{aligned}$$

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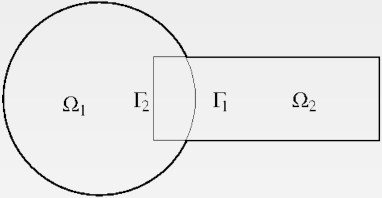


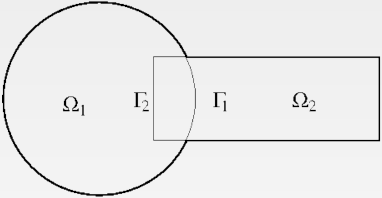
$$\begin{aligned} \Delta u_1^n &= 0 && \text{in } \Omega_1 & \Delta u_2^n &= 0 && \text{in } \Omega_2 \\ u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 & u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 & u_2^n &= u_1^n && \text{on } \Gamma_2 \end{aligned}$$

- P.L. Lions (1988) (parallel)

$$\begin{aligned} \Delta u_1^n &= 0 && \text{in } \Omega_1 & \Delta u_2^n &= 0 && \text{in } \Omega_2 \\ u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 & u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 & u_2^n &= u_1^{n-1} && \text{on } \Gamma_2 \end{aligned}$$

Bibliography

Evolution problem



Original Schwarz Method :

- At each t time, solutions $\Rightarrow$ interface conditions, $\Downarrow$
- We must use the same time step for each subdomains. $\Downarrow$

Classical Schwarz Waveform Relaxation :

- Solve with different time steps, $\Uparrow$
- interface conditions for the time interval. $\Uparrow$

Bibliography

Classical Schwarz Waveform Relaxation
wave equation
with overlap
"Dirichlet" interface conditions :

$$\Box_c u = \partial_t^2 u - c^2 \Delta u$$

$$\begin{aligned} \Box_c u_1^n &= 0 && \text{in } \Omega_1 \times [0, T] \\ u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 \\ u_1^n|_{t=0} &= u_0 && \text{in } \Omega_1 \\ \dot{u}_1^n|_{t=0} &= u_1 && \text{in } \Omega_1 \end{aligned} \qquad \begin{aligned} \Box_c u_2^n &= 0 && \text{in } \Omega_2 \times [0, T] \\ u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_2^n &= u_1^{n-1} && \text{on } \Gamma_2 \\ u_2^n|_{t=0} &= u_0 && \text{in } \Omega_2 \\ \dot{u}_2^n|_{t=0} &= u_1 && \text{in } \Omega_2 \end{aligned}$$

Finite step convergence[1] : $n > \dfrac{T\,c}{\delta}$,

δ : overlap.

[1] Gander-Halpern (2004)

Bibliography

Different interface conditions :

- Neumann,
- Robin, ...

Replace "$u_i^n = u_j^{n-1}$" by "$B_i u_i^n = B_i u_j^{n-1}$" on Γ_i.

$$\begin{array}{rll} \Box_c u_1^n = 0 & \text{in } \Omega_1 \times [0,T] \\ u_1^n = g & \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ B_1 u_1^n = B_1 u_2^{n-1} & \text{on } \Gamma_1 \\ u_1^n|_{t=0} = u_0 & \text{in } \Omega_1 \\ \dot{u}_1^n|_{t=0} = u_1 & \text{in } \Omega_1 \end{array} \qquad \begin{array}{rll} \Box_c u_2^n = 0 & \text{in } \Omega_2 \times [0,T] \\ u_2^n = g & \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ B_2 u_2^n = B_2 u_1^{n-1} & \text{on } \Gamma_2 \\ u_2^n|_{t=0} = u_0 & \text{in } \Omega_2 \\ \dot{u}_2^n|_{t=0} = u_1 & \text{in } \Omega_2 \end{array}$$

Bibliography

Error variables " $u_i^n - u_i^{n-1} \to u_i^n$ " :

$$
\begin{array}{rll}
\Box_c u_1^n = 0 & \text{in } \Omega_1 \times [0, T] \\
u_1^n = 0 & \text{on } \partial\Omega \cap \overline{\Omega}_1 \\
B_1 u_1^n = B_1 u_2^{n-1} & \text{on } \Gamma_1 \\
u_1^n|_{t=0} = 0 & \text{in } \Omega_1 \\
\dot{u}_1^n|_{t=0} = 0 & \text{in } \Omega_1
\end{array}
\qquad
\begin{array}{rll}
\Box_c u_2^n = 0 & \text{in } \Omega_2 \times [0, T] \\
u_2^n = 0 & \text{on } \partial\Omega \cap \overline{\Omega}_2 \\
B_2 u_2^n = B_2 u_1^{n-1} & \text{on } \Gamma_2 \\
u_2^n|_{t=0} = 0 & \text{in } \Omega_2 \\
\dot{u}_2^n|_{t=0} = 0 & \text{in } \Omega_2
\end{array}
$$

Can you determine " $B_i, i = 1, 2$ " as :

$$
\left\{ \begin{array}{l} B_1 u_2^{n-1} \neq 0 \\ B_2 u_1^{n-1} \neq 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} B_1 u_2^n = 0 \\ B_2 u_1^n = 0 \end{array} \right. ?
$$

Bibliography

With the "Dirichlet to Neumann" operator DtN_1 :

$$DtN_1(v) : v \text{ on } \Gamma_1 \mapsto \frac{\partial u}{\partial \nu_1} \text{ on } \Gamma_1$$

$$\begin{aligned} \Box_c u &= 0 \quad \text{in } \Omega_1 \times [0, T] \\ u &= 0 \quad \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ u &= v \quad \text{on } \Gamma_1 \\ u|_{t=0} &= 0 \quad \text{in } \Omega_1 \\ \dot{u}|_{t=0} &= 0 \quad \text{in } \Omega_1 \end{aligned}$$

With $\Gamma_2 = \Gamma_1$, we get (conservative flow and $u = v$ on Γ_1) :

$$\frac{\partial u}{\partial \nu_2} = -\frac{\partial u}{\partial \nu_1} = -DtN_1(v) = -DtN_1(u)$$

so we get : $\dfrac{\partial u}{\partial \nu_2} + DtN_1(u) = B_2(u) = 0$ on Γ_2.

("transparent" operator)

Bibliography

With these new B_i :

$$\begin{array}{rll} \Box_c u_1^n = 0 & \text{in } \Omega_1 \times [0,T] \\ u_1^n = 0 & \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ B_1 u_1^n = B_1 u_2^{n-1} & \text{on } \Gamma_1 \\ u_1^n|_{t=0} = 0 & \text{in } \Omega_1 \\ \dot{u}_1^n|_{t=0} = 0 & \text{in } \Omega_1 \end{array} \qquad \begin{array}{rll} \Box_c u_2^n = 0 & \text{in } \Omega_2 \times [0,T] \\ u_2^n = 0 & \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ B_2 u_2^n = B_2 u_1^{n-1} & \text{sur } \Gamma_2 \\ u_2^n|_{t=0} = 0 & \text{in } \Omega_2 \\ \dot{u}_2^n|_{t=0} = 0 & \text{in } \Omega_2 \end{array}$$

So with $u_i^n \neq 0, i = 1, 2$, we get:

$$\Rightarrow B_i u_j^n = 0, i \neq j,$$

and : $u_i^{n+1} = 0, i = 1, 2$ (we get the convergence).

Bibliography

Classical Schwarz Waveform Relaxation
wave equation
without overlap
"transparent" operator :

$$\square_c u = \partial_t^2 u - c^2 \Delta u$$

$$\begin{array}{llll}
\square_c u_1^n = 0 & \text{in } \Omega_1 \times [0,T] & \square_c u_2^n = 0 & \text{in } \Omega_2 \times [0,T] \\
u_1^n = 0 & \text{on } \partial\Omega \cap \overline{\Omega}_1 & u_2^n = 0 & \text{on } \partial\Omega \cap \overline{\Omega}_2 \\
B_1 u_1^n = B_1 u_2^{n-1} & \text{on } \Gamma_1 & B_2 u_2^n = B_2 u_1^{n-1} & \text{on } \Gamma_2 \\
u_1^n|_{t=0} = 0 & \text{in } \Omega_1 & u_2^n|_{t=0} = 0 & \text{in } \Omega_2 \\
\dot{u}_1^n|_{t=0} = 0 & \text{in } \Omega_1 & \dot{u}_2^n|_{t=0} = 0 & \text{in } \Omega_2
\end{array}$$

Optimal convergence (only 2 steps) [1].

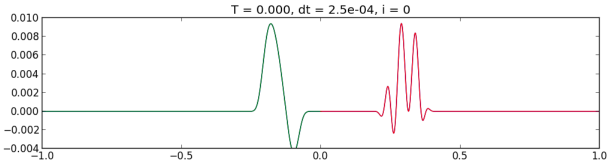
[1] Gander-Halpern-Nataf (2003), Gander-Halpern (2004)

Wave equation

"Transparent" operator :

- 1D $\Rightarrow$ with Engquist-Majda (order 1),
- 2-3D $\Rightarrow$ Engquist-Majda, Higdon (order > 1),

Example 1D : step 0

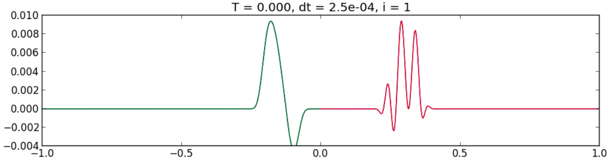


Wave equation

"Transparent" operator :

- 1D $\Rightarrow$ with Engquist-Majda (order 1),
- 2-3D $\Rightarrow$ Engquist-Majda, Higdon (order > 1),

Example 1D : step 1

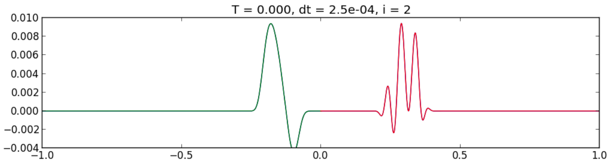


Wave equation

"Transparent" operator :

- 1D $\Rightarrow$ with Engquist-Majda (order 1),
- 2-3D $\Rightarrow$ Engquist-Majda, Higdon (order > 1),

Example 1D : step 2

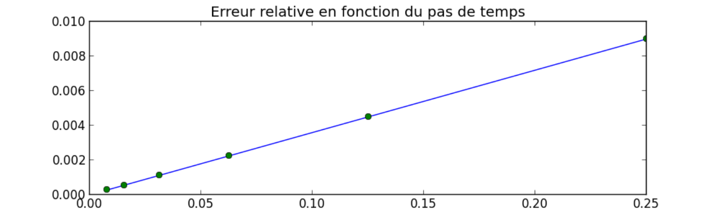


Wave equation

"Transparent" operator :

- 1D $\Rightarrow$ with Engquist-Majda (order 1),
- 2-3D $\Rightarrow$ Engquist-Majda, Higdon (order > 1),

Example 1D :



Hydro. wave model H.W. A.B.C.(1/2)

The boundary condition "$B_1 u_1 = B_1 u_2$" :

$$\left[\prod_{j=1}^{p}(a_j\partial_t + c\partial_\nu)^2\right](a_0\partial_t + c\partial_\nu)u_1 = \left[\prod_{j=1}^{p}(a_j\partial_t + c\partial_\nu)^2\right](a_0\partial_t + c\partial_\nu)u_2$$

or :

$$\left[\prod_{j=1}^{p}(a_j\partial_t + c\partial_\nu)^2\right]\underbrace{(a_0\partial_t + c\partial_\nu)(u_1 - u_2)}_{a_0\partial_t\phi_1} = 0$$

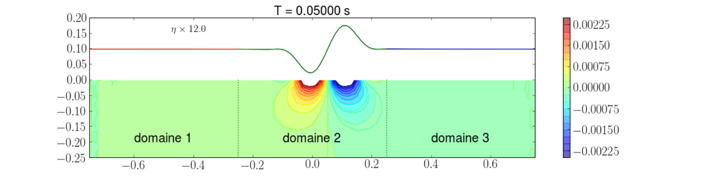
so we get :

$$\begin{aligned}
(a_0\partial_t + c\partial_\nu)u_1 &= a_0\partial_t\phi_1 + (a_0\partial_t + c\partial_\nu)u_2,\\
(a_1\partial_t + c\partial_\nu)\phi_1 &= (a_1\partial_t - c\partial_\nu)\phi_2,\\
&\vdots\\
(a_j\partial_t + c\partial_\nu)\phi_j &= (a_j\partial_t - c\partial_\nu)\phi_{j+1},\\
&\vdots\\
\phi_{p+1} &= 0.
\end{aligned}$$

Hydro. wave model H.W. A.B.C.(2/2)

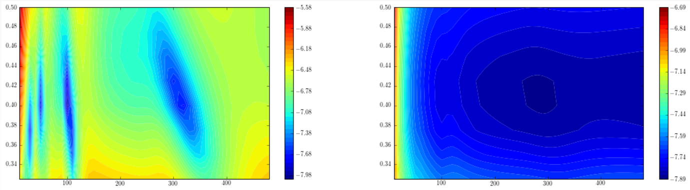
Coupled model :

($c_f = 1000$, $J = 8$, $a = 0.4\ c_f / C_{min} - 320\ c_f / C_{min}$, $pc_{dtc} = 0.75$)
($nsom_s = 41$, $nsom_f = 820$ by subdomain)



J quadratic differences on the interface :

(left (surface), right (basin), $J(a_{min}, a_{max})$)



Conclusion

- to identify a_i minimize the convergence rate (Schwarz method),
- a small overlap (for evanescents waves),
- with flow, 3D,
- ...