Asymptotic Derivation of Shallow Water equations for Non Newtonian Free Surface Flow

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Introduction : Non Newtonian Flows I

Glucose

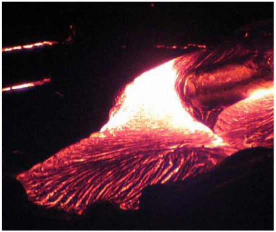
Carbopol

Concret

torential Lava I

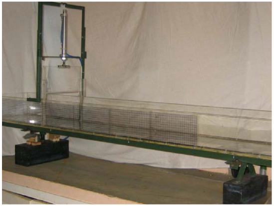
torential Lava II

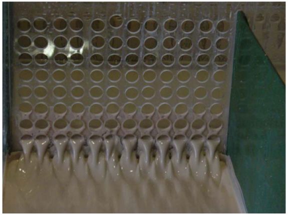
Powder Snow Avalanche



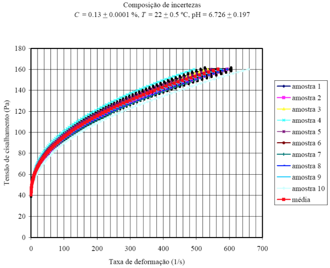
volcanic Lava

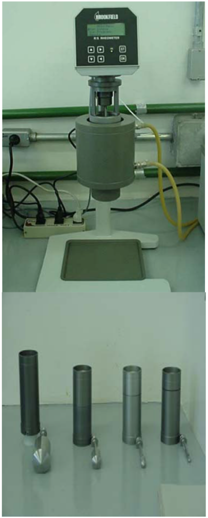
Introduction :Non Newtonian Flows II






Introduction : Non Newtonian Rheology





Carbopol $0.4 < n < 1$, Kaolin slurry $0.2 < n < 0.4$

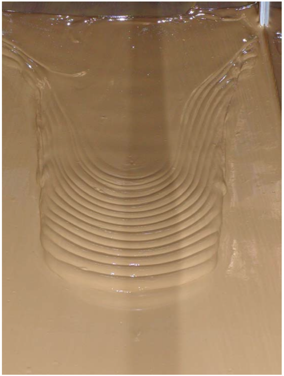
Typical Rheology

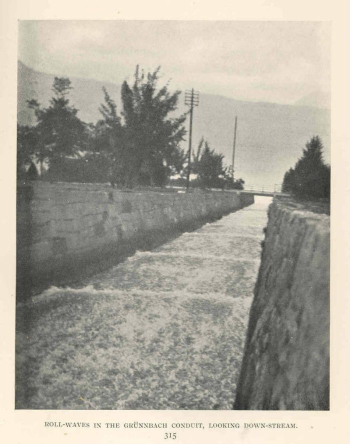
Power Law : $\tau = \mu |D(u)|^{n-1} D(u)$,

Herschel-Bulkley (Bingham $n = 1$)

$$\mu |D(u)|^{n-1} D(u) = \begin{cases} 0 \; if \; |\tau| < \sigma_0 \\ \tau - \frac{\sigma_0}{|D(u)|} D(u) \; if \; |\tau| \geq \sigma_0 \end{cases}$$

Introduction: Free surface instabilities





Left: Bentonite suspension (From Chanson-Coussot 2004)
Right: planar (turbulent) roll-waves (Stoker 1949)

Introduction: laminar roll waves in laboratory

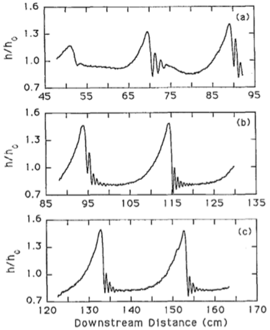


FIG. 3. The evolution of solitary waves forced at $f=1.5$ Hz with $\beta=6.4°$, $R=29$, and Weber number $W=35$. Three wave profiles are measured at increasing distances from the inlet to show the spatial evolution. Phase-sensitive averaging is employed here to reduce imaging noise.

Liu and Gollub experience
(Phys of Fluids 94)

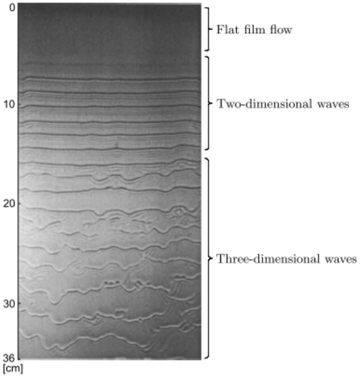
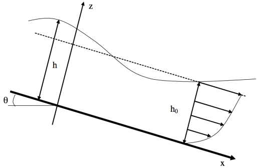


Photo of 2-d roll-waves
(Park et Nosoko AIChE, 2003)

- General model: Navier-Stokes (NS) equations with a free surface ,
  Unknowns: velocity $\vec{u} = (u, w) \in \mathbb{R}^2$, pressure p,
  fluid domain $\Omega_t = \{(x, z), x \in \mathbb{R}^n, 0 \leq z \leq h(x, t)\}$



- **Methodology**: under suitable assumptions, derive simpler models
  - **Aspect ratio**: $\varepsilon = H/L$, (characteristic fluid height/characteristic horizontal wavelengthl).
  - **Reynolds Number**: $Re = \rho H^n U^{2-n}/\mu$
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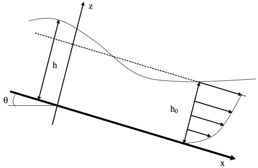

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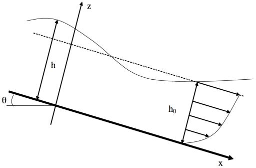

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Asymptotic Models. Long Wave expansion / Thin Films I

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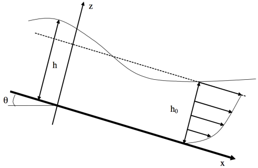

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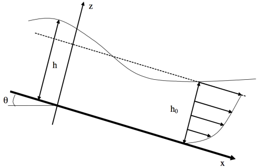

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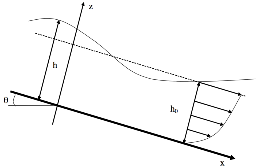

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Main hypothesis: $\varepsilon = H/L \ll 1$ (thin films).

Gravity Driven Flow : Viscous Shear balance gravity and gives the characteristic scale of the flow –> Parabolic velocity profile (Newtonian case)

$$\int_0^h u^2 = \frac{6q^2}{5h} + O(\varepsilon)$$
$$\partial_t h + \partial_x q = 0,$$
$$\partial_t q + \partial_x \Big(\int_0^h u^2 + \frac{p}{F^2} dz\Big) = \frac{1}{\varepsilon R_e}(\lambda h - \tau_{xz}|_{z=0}) + \partial_x \Big(\int_0^h \tau_{xx} dz\Big).$$

- Closure requires knowledge of $\int_0^h u^2$ and $\int_0^h \frac{p}{F^2}$ at $O(\varepsilon)$, and $\tau_{xz}|_{z=0}$ at $O(\varepsilon^2)$ ($o(\varepsilon)$ at least) (except if $\lambda = \frac{R_e}{F^2}\sin\theta = 0$, or $\tau_{xz} = O(\varepsilon)$
- **Remark:** $Re\sin(\theta) = \lambda F^2, \quad \lambda = 3$ if we take uniform flow as reference
- With classical wall condition $\tau_{xz}|_{z=0} = \lambda h + O(\varepsilon) = \frac{3q}{h^2} + O(\varepsilon)$, Friction condition $\tau_{xz} = ku$ idem unless $k = O(\varepsilon)$–> Asymptotic studied by Gerbeau Perthame (2001),

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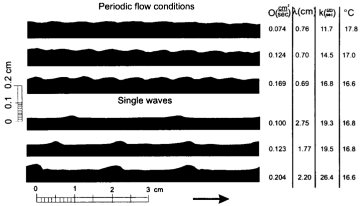
Kapitza (1949)



- Benney (1963)–> One equation model first order accurate (Lubrication approximation)

$$h_t + q_x = 0$$

$$q = \frac{\lambda h^3}{3} + \varepsilon R_e \left(\frac{2}{15} \lambda^2 h^6 \frac{\partial h}{\partial x} + \left(\frac{1}{3} h^3 \frac{\varepsilon^2}{We} \frac{\partial^3 h}{\partial x^3} - \frac{1}{3} h^3 \frac{\cos\theta}{F^2} \frac{\partial h}{\partial x} \right) \right)$$

Viscous Thin Films II

- Two equations (Shallow Water type) Shkadov (1969) –> unaccurate (Stability threshold wrong : Von Karmann, integral BL approach) Ruyer-Quil, Manneville (1998)–> first order accurate, 2nd order accurate (2000) (Weighted Residual technics)

$$\partial_t h + \partial_x q = 0$$

$$\partial_t q + \frac{17}{7}\frac{q\partial_x q}{h} - \frac{9}{7}\frac{q^2}{h^2}\partial_x h + \frac{5}{6R_e}\lambda\mathrm{cotan}(\theta)h\partial_x h = \frac{5}{\varepsilon 6R_e}\left(\lambda h - (\frac{3q}{h^2})\right)$$

- Conservation form (2006,...,2011) BCNV

$$q_t + \frac{\partial}{\partial x}\left(\frac{q^2}{h} + p\,(h)\right) - \frac{A_1\varepsilon^2}{We}hh_{xxx} = \frac{A_1}{\beta}\left(\lambda h - 3\frac{q}{h^2}\right)$$

$$p(h) = \left(\frac{4}{45} - \frac{2A_1}{25}\right)h^5\lambda^2 + \frac{\lambda A_1}{2R_e}h^2\cot\theta$$

- Mathematical complete justification : (Bresch, Noble (2007))

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Asymptotic description of Power Law fluids I

Lubrification models on h (Ng, Mei 1994)

$$\partial_t h + \partial_x \left(\frac{nh^2(\lambda h)^{1/n}}{2n+1} \right) = \varepsilon \partial_x \left(\left(\frac{\lambda^{1/n} h^{2+1/n} \mathrm{cotan}(\theta)}{2n+1} - \frac{2\lambda^{3/n-1} h^{3+3/n} R_e}{(2n+1)(3n+2)} \right) \partial_x h \right)$$

- Method: expand the velocity field up to order 1 w.r.t. ε and plug this expansion into the conservation law $\partial_t h + \partial_x \int_0^h u(.,z)dz = 0$
- Llinear stability (necessary condition):

$$Re < Re_c = \frac{3n+2}{2} \lambda^{1-2/n} \mathrm{cotan}(\theta).$$

- Problem: if $Re > Re_c$, the lubrication equation is locally ill posed.

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Asymptotic description of Power Law fluids II

First order consistent model (E.D. Fernandez-Nieto, P.N., J.-P. Vila, 2010,JNN), $n < 1$

$$
\begin{aligned}
&\partial_t h + \partial_x q = 0, \\
&\partial_t q + \partial_x \Big(\frac{q^2}{h} + \frac{n^2 \lambda^{2/n} h^{3+2/n}}{(3n+2)(2n+1)^2} + \frac{(3n+2)\lambda \mathrm{cotan}(\theta) h^2}{4(2n+1)R_e} \Big) \\
&\qquad = \frac{3n+2}{2\varepsilon(2n+1)R_e} \left(\lambda h - \big(\frac{(2n+1)q}{n\, h^2}\big)^n \right). \qquad (1)
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see also (Amaouche et al, 2009,CRAS) for a non conservation form via Weighted Residual

Remark:

- **Consequence:** first order models are not unique
- **Uniqueness:** conservativity of the system, Galilean invariance

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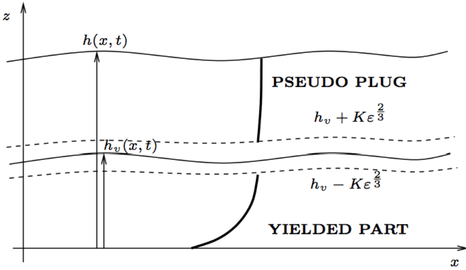
In non dimensionnal form

$$D(u) = \begin{cases} 0 \text{ if } |\tau| < B \\ \tau - \frac{\sigma_0}{|D(u)|} D(u) \text{ if } |\tau| \geq B \end{cases}$$



Reference Flow

$$h < h_c, \quad u = w = 0,$$

$$h > h_c, \quad u(z) = \lambda((h - h_c)z - \frac{z^2}{2}), \ \forall z \in (0, \ h - h_c = h_v),$$

$$u(z) = \lambda \frac{h_v^2}{2}, \ \forall z \in (h_v, \ h).$$

Shallow water models for HB/Bingham fluids

First order consistent model (Nieto,Noble,V., 2011, JNN):

$$\partial_t h + \partial_x q = 0,$$

$$\partial_t q + \partial_x \Big(\Gamma_1(h) \frac{q^2}{h} + \frac{c\,h^2}{2\,F^2} + \Gamma_2(h) \Big) = \frac{1}{\varepsilon R_e} (\lambda h - B - 6\lambda \frac{q}{2\lambda h_v^2 + 3B\,h_v})$$

$$+ \frac{3\pi B^2 |\partial_x h|}{R_e\,(2\lambda h_v + 3B)}$$

$$\int_0^h u^2 = \Gamma_1(h) \frac{q^2}{h}, \quad \Gamma_1(h) = \frac{h}{(h+\frac{B}{2\lambda})^2} \big(\frac{6}{5} h + (\frac{9}{4} - \frac{6}{5}) \frac{B}{\lambda} \big).$$

$$\Gamma_2(h) = \frac{\delta c}{2}\, h_c\, (h + \frac{3\,h_c}{2} \log(\frac{h_v}{3} + \frac{h_c}{2})) - \lambda^2 \Big(\frac{h_v^5}{75} + \frac{h_c\,h_v^4}{10} + \frac{3h_c^2\,h_v^3}{10} \Big)$$

$$-\lambda^2 \Big(\frac{3h_v^2 h_c^3}{40} - \frac{9h_v h_c^4}{40} + \frac{27h_c^5}{80} \log(\frac{h_v}{3} + \frac{h_c}{2}) \Big).$$

This model justify an heuristic derivation of Piau (J Rheology 1999)

Shallow water models: weak form of (NS) eqs

- Write Navier-Stokes equations in the scaling $\varepsilon = H/L \ll 1$

$$\tau_{xz} = \lambda(h-z) + \varepsilon\partial_x\left(\int_z^h \tau_{xx} - \frac{R_e p}{F^2} d\tilde{z}\right)$$
$$- \varepsilon R_e \left(\partial_t(\int_z^h u\, d\tilde{z}) + \partial_x(\int_z^h u^2\, d\tilde{z}) - uw\right),$$
$$\tau_{xx} + \frac{R_e p}{F^2} = \lambda \mathrm{cotan}(\theta)(h-z) - \varepsilon\partial_x(\int_z^h \tau_{xz}\, d\tilde{z})$$
$$+ \varepsilon^2 R_e \left(\partial_t(\int_z^h w\, d\tilde{z}) + \partial_x(\int_z^h uw\, d\tilde{z}) - w^2\right)$$
$$w = \partial_t h + \partial_x\left(\int_z^h u d\tilde{z}\right).$$

- Solution in the sense of distribution: $u, w, p, \tau \in L^1_z(0,h)$, $u, w \in L^2_z(0,h)$ (not necessarily bounded).

Shallow water models: constitutive law

- Invert the constitutive law

$$2\varepsilon\partial_x u = \left(\tau_{xx}^2 + \tau_{xz}^2\right)^{\dfrac{1-n}{2n}} \tau_{xx},$$

$$\partial_z u + \varepsilon^2\partial_x w = \left(\tau_{xx}^2 + \tau_{xz}^2\right)^{\dfrac{1-n}{2n}} \tau_{xz}.$$

- Basic flow (collect $O(\varepsilon^0)$ terms)

Basic flow

$$u^{(0)} = \frac{n\lambda^{1/n}}{n+1}\left(h(x,t)^{1+1/n} - (h(x,t)-z)^{1+1/n}\right),$$

$$\frac{R_e p^{(0)}}{F^2} = \lambda \mathrm{cotan}(\theta)\,(h(x,t)-z)\,, \quad \tau_{xz}^{(0)} = \lambda(h(x,t)-z), \quad \tau_{xx}^{(0)} = 0.$$

Shallow water models: method of derivation

- Integration of divergence free condition and horizontal momentum equation on $(0, h)$

Integrated free divergence and horizontal momentum equations

$$\partial_t h + \partial_x q = 0,$$
$$\partial_t q + \partial_x \Big(\int_0^h u^2 + \frac{p}{F^2} dz \Big) = \frac{1}{\varepsilon R_e} (\lambda h - \tau_{xz}|_{z=0}) + \partial_x \Big(\int_0^h \tau_{xx} dz \Big).$$

- "Expand" (in L^1 sense!) (u, w, p, τ) with respect to ε at order 2.
- Expand $\int_0^h u^2$, $\int_0^h p$, $\tau_{xz}|_{z=0}$, $\int_0^h \tau_{xx}$ w.r.t $\varepsilon \ll 1$ (coefficients depend on h and its derivatives)
- Write a closure with h, q to mimic a shallow water model

Shallow water models: order 2 expansion

- Up to order one, no restriction on $n > 0$ BUT

$$\tau_{xx}^{(1)} = 2\lambda \left(h^{1/n}(h-z)^{1-1/n} - (h-z) \right) \partial_x h.$$

- $\tau_{xx}^{(1)}$ is integrable only if $n > 1/2$.
- Collect $O(\varepsilon^2)$ terms

$$\begin{aligned}
\tau_{xz}^{(2)}|_{z=0} = {} & R_e^2 \left(\frac{2\,\lambda^{\frac{4}{n}-1}\,h^{\frac{4}{n}+4}\,\left(5\,n^2+14\,n+6\right)}{(2\,n+1)^2\,(3\,n+2)\,(4\,n+3)} \right) \partial_{xx} h \\
& - R_e \mathrm{cotan}(\theta) \left(\frac{2\,\lambda^{\frac{2}{n}}\,h^{\frac{2}{n}+3}\,(n+2)}{(2\,n+1)\,(3\,n+2)} \right) \partial_{xx} h + \frac{\lambda\,h^2\,(2\,n+3)}{2\,(2\,n-1)} \partial_{xx} h \\
& + R_e^2 \left(\frac{2\,\lambda^{\frac{4}{n}-1}\,h^{\frac{4}{n}+3}\,(n+1)\,\left(12\,n^2+46\,n+21\right)}{n\,(2\,n+1)^2\,(3\,n+2)\,(4\,n+3)} \right) (\partial_x h)^2 \\
& - R_e \mathrm{cotan}(\theta) \left(\frac{\lambda^{\frac{2}{n}}\,h^{\frac{2}{n}+2}\,(n+3)}{n\,(2\,n+1)} \right) (\partial_x h)^2 + \frac{\lambda\,h\,(2\,n+3)}{2\,n-1} (\partial_x h)^2.
\end{aligned}$$

Shallow water models: order 2 expansion

- $u^{(2)} \sim f(h, \partial_x h)(h - z)^{1-1/n}$ unbounded but integrable if $n > 1/2$.

$$
\begin{aligned}
q^{(2)} = {} & R_e^2 \left(\frac{4\lambda^{\frac{5}{n}-2}\, h^{\frac{5}{n}+5}\, \left(8n^2 + 25n + 12\right)}{(2\,n+1)^2\, (3\,n+2)\, (4\,n+3)\, (5\,n+4)} \right) \partial_{xx} h \\
& - R_e \mathrm{cotan}(\theta) \left(\frac{2\lambda^{\frac{3}{n}-1}\, h^{\frac{3}{n}+4}\, \left(5n^2 + 14n + 6\right)}{(2\,n+1)^2\, (3\,n+2)\, (4\,n+3)} \right) \partial_{xx} h \\
& + \frac{4\lambda^{\frac{1}{n}}\, h^{\frac{1}{n}+3}\, (n+2)}{3\,(2\,n-1)\,(3\,n+1)} \partial_{xx} h \\
& + R_e^2 \left(\frac{3\,\lambda^{\frac{5}{n}-2}\, h^{\frac{5}{n}+4}\, (n+1)\, \left(35\,n^2 + 226\,n + 120\right)}{2\,n\,(2\,n+1)^2\, (3\,n+2)\, (4\,n+3)\, (5\,n+4)} \right) (\partial_x h)^2 \\
& - R_e \mathrm{cotan}(\theta) \left(\frac{8\,\lambda^{\frac{3}{n}-1}\, h^{\frac{3}{n}+3}}{n\,(2\,n+1)\,(3\,n+2)} \right) (\partial_x h)^2 \\
& - \frac{\lambda^{\frac{1}{n}}\, h^{\frac{1}{n}+2}\, \left(10n^2 + 25n + 7 - 3(n-1)(2n-1)\mathrm{cotan}^2(\theta)\right)}{6\,n\,(2\,n-1)\,(2\,n+1)} (\partial_x h)^2 .
\end{aligned}
$$

Shallow water models: form of the model

- The first order model is, under the assumption of conservatively and Galilean, invariance unique.
- Second order model: conservative form? **Galilean invariance**

We assume the following form of the shallow water model:

Form of the second order consistent model $n > 1/2$

$$
\begin{aligned}
&\partial_t h + \partial_x (h\bar{u}) = 0,\\
&\partial_t h\bar{u} + \partial_x \Big(h\bar{u}^2 + \frac{n^2 \lambda^{2/n} h^{3+2/n}}{(3n+2)(2n+1)^2} + \frac{(3n+2)\lambda \mathrm{cotan}(\theta) h^2}{4(2n+1)R_e} \Big)\\
&\qquad = \frac{3n+2}{2\varepsilon(2n+1)R_e} \Big(\lambda h - (\frac{(2n+1)\bar{u}}{n\, h})^n \Big)\\
&\qquad + \varepsilon \left(\partial_x (A(h,\bar{u},n)\partial_x q) + B(h,n)\partial_{xx} h + C(h,n)(\partial_x h)^2 \right)
\end{aligned}
$$

for suitable choice of A, B, C to obtain a second order model.

Shallow water equations: form of the model (remarks)

$u = U(z, h, \partial_x h, \cdots, \varepsilon)$ —> $q = F(h, \partial_x h, \cdots, \varepsilon)$ —> an ∞ number of models. Why a 2 moments SW model since q is not a free unknown???

Back to a shallow water type model (heuristic)

$$\partial_t h + \partial_x q = 0, \quad \partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{h^2}{2F^2} \right) = \frac{1}{\varepsilon} \left(h - \frac{q}{h^2} \right) + \varepsilon \partial_{xx} q$$

- If $F > 2$, constant states are unstable, nonlinear wave so called roll-waves occur that are nonlinearly stable (Noble, Rodrigues, Johnson, Zumbrun, '11)
- $\varepsilon \to 0$, $F < 2$: relaxation $q = f(h, \partial_x h, \varepsilon)$ and a lubrication equation is satisfied. Conclusion (heuristic): all formal computations can be justified there and fluid pressure/velocity are function of h
- $\varepsilon \to 0$, $F > 2$: asymptotic is governed by 2 moments Whitham's modulation equations (Noble, Rodrigues '11). The formal assumption that q function of h is in general untrue. Conclusion (heuristic): a real two moment expansion of fluid pressure/velocity is needed

Conclusion and perspectives

1. Condition $n > 1/2$: well posedness of the (linearized) Navier Stokes equations?
2. If $n < 1/2$: ill posedness? effect of a regularization?
3. Other boundary conditions at the bottom: slip conditions? (application to flow down porous inclined plane)
4. Other "singular" constitutive laws: Bingham law, Herschel Bulkley law: order 2 theory?