# Modeling and numerics for (shallow) (water) flows Part 2 : Beyond the Saint-Venant system

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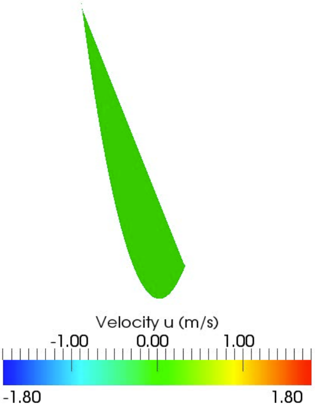
# Beginning of the story

- Saint-Venant system with sources

$$\begin{aligned} \partial_t h + \partial_x hu &= 0, \\ \partial_t hu + \partial_x \left(hu^2 + gh^2/2\right) &= -gh\,\partial_x b \end{aligned}$$

- Numerics: Finite volume method
  - Positive (and entropic) scheme
  - Well-balanced scheme
- Softwares
  MASCARET, TELEMAC 2D (EDF-LNHE)
  http://www.opentelemac.org/index.php/presentation?id=17
  FULLSWOF (Univ. Orléans)
  http://www.univ-orleans.fr/mapmo/soft/FullSWOF/

# A discriminant test case ?

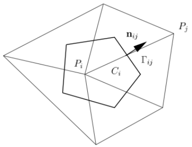


# Joint work with

- ANGE group
  J. Sainte-Marie, M.O. Bristeau,
  Y. Penel, P. Ung, N. Seguin, E. Godlewski
- EDF LNHE – Saint-Venant Lab.
  N. Goutal, M. Jodeau, S. Boyaval
- Academic partners
  F. Bouchut, C. Berthon, C. Chalons, O. Delestre, S. Cordier...

# Finite Volume Method

$$\partial_t u + \nabla \cdot f(u) = 0, \qquad u \in \mathbb{R}^p, \quad f : \mathbb{R}^p \to \mathbb{R}^{2\times p}$$




- Integration on the prism $C_i \times [t^n, t^{n+1}]$

$$U_i^{n+1} = U_i^n - \sum_{j \in V(i)} \sigma_{ij}^n F(U_i^n, U_j^n, n_{ij})$$

with

$$\sigma_{ij}^n = \frac{\Delta t^n |\Gamma_{ij}|}{|C_i|}, \qquad F(U_i^n, U_j^n, n_{ij}) \approx \int_{\Gamma_{ij}} \int_{t^n}^{t^{n+1}} f(U(t,s)).n_{ij} dtds$$

# Relaxation solver

- Godunov scheme
  - Consistency, Stability
  - Complexity : Solution of the Riemann problem
- Relaxation models : Introduction of a larger system
  - that is hyperbolic (with LD fields)
  - that formally approaches the original ones
  - that ensures some stability property
  - for which the (homogeneous) Riemann problem is easy to solve
- Numerical algorithm
  - Definition of auxiliary variables from physical ones
  - Solution of the Riemann problem
  - Computation of the physical variables at the next time step

(Chen et al. [94], Jin-Xin [95], Bouchut [04]...)

# Relaxation solver

- Example for scalar conservation law
  - Original SCL
$$\partial_t u + \partial_x f(u) = 0$$
  - Relaxation model
$$\begin{aligned} \partial_t u + \partial_x v &= 0 \\ \partial_t v + c^2 \partial_x u &= \frac{1}{\epsilon}(f(u) - v) \end{aligned}$$
  - Stability requirement
$$c \geq |f'(u)|$$
  - One retrieves the Rusanov scheme (i.e. HLL scheme with $c_l = -c_r$)

# Suliciu relaxation solver : No source

- Suliciu relaxation model for SW system

$$
\begin{aligned}
\partial_t h + \partial_x h\bar{u} &= 0 \\
\partial_t h\bar{u} + \partial_x \left(h\bar{u}^2 + \pi\right) &= 0 \\
\partial_t h\pi + \partial_x(h\pi\bar{u}) + c^2\partial_x \bar{u} &= \frac{1}{\epsilon}\left(\frac{gh^2}{2} - \pi\right)
\end{aligned}
$$

- Stability criterion

$$c \geq h\sqrt{gh}$$

- Eigenvalues (distincts and associated to LD fields)

$$\lambda_\pm = u \pm \frac{c}{h}, \quad \lambda_u = u$$

(Suliciu [90,92])

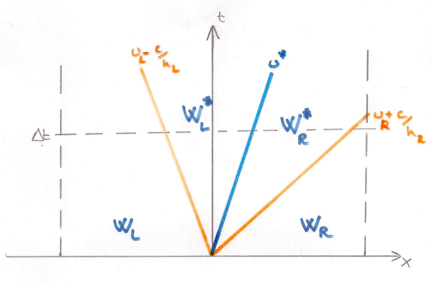
# Suliciu relaxation solver : No source

- Initialization : Relaxation source term with "$\epsilon = 0$"

$$\pi_i^n = g(h_i^n)^2/2$$

- Solution of the (homogeneous) Riemann problem

$$w^R_{[W_l,W_r]}(t,x), \quad W_=(h,\bar{u},\pi)^T$$



# Suliciu relaxation solver : No source

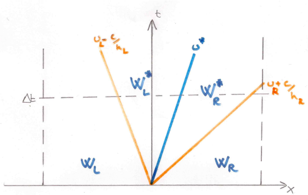
- Computation of the intermediate states

$$u^* = u_M - \frac{1}{2c}\Delta\pi, \qquad \pi^* = \pi_M - \frac{c}{2}\Delta u$$

$$\tau_l^* = \tau_l + \frac{1}{2c}\Delta u - \frac{1}{2c^2}\Delta\pi, \quad \tau_r^* = \tau_r + \frac{1}{2c}\Delta u + \frac{1}{2c^2}\Delta\pi, \quad "\tau = \frac{1}{h}"$$

- Waves celerities

$$\lambda_l = u_l - \frac{c}{h_l}, \quad \lambda_u = u^*, \quad \lambda_+ = u_r + \frac{c}{h_r}, \qquad "c \geq h\sqrt{gh}"$$




Extension to vacuum : Bouchut, "Nonlinear Stability of FVM for HCL", Birkhäuser, 2004

# Suliciu relaxation scheme : With source

- First version

$$\begin{aligned}
\partial_t h + \partial_x h\bar{u} &= 0\\
\partial_t h\bar{u} + \partial_x \left(h\bar{u}^2 + \pi\right) + gh\partial_x z &= 0\\
\partial_t h\pi + \partial_x(h\pi\bar{u}) + c^2\partial_x\bar{u} &= \frac{1}{\epsilon}\left(\frac{gh^2}{2} - \pi\right)\\
\partial_t z &= 0
\end{aligned}$$

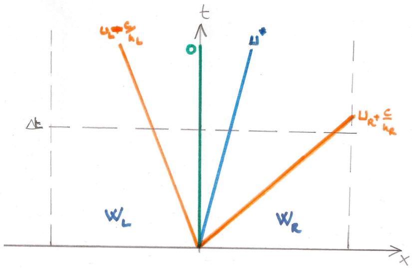
- Eigenvalues (associated to LD fields)

$$\lambda_\pm = u \pm \frac{c}{h}, \quad \lambda_u = u, \quad \lambda_0 = 0$$

(Bouchut [04])

# Suliciu relaxation solver : With source

- Computation of the new solution
  Non conservative system ⇝ Integration process (no flux)
- Properties
  Consistency, WB, Positivity, Entropy inequality
- Effective computation
  Eigenvalues not ordered ⇝ Multiple Riemann pbs to solve



# Suliciu relaxation solver : With source

- Alternative version

$$
\begin{aligned}
\partial_t h + \partial_x h\bar{u} &= 0 \\
\partial_t h\bar{u} + \partial_x \left(h\bar{u}^2 + \pi\right) + gh\partial_x \tilde{z} &= 0 \\
\partial_t h\pi + \partial_x (h\pi\bar{u}) + c^2 \partial_x \bar{u} &= \frac{1}{\epsilon}\left(\frac{gh^2}{2} - \pi\right) \\
\partial_t \tilde{z} + \bar{u}\partial_x \tilde{z} &= \frac{1}{\epsilon}(z - \tilde{z})
\end{aligned}
$$

- Eigenvalues (associated to LD fields)

$$
\lambda_\pm = u \pm \frac{c}{h}, \quad \lambda_u = u \text{ (double)}
$$

Riemann problem is under-determined...

# Suliciu relaxation solver : With source

- Linearized alternative version

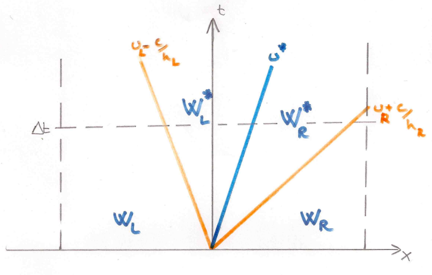
$$\begin{aligned}
\partial_t h + \partial_x h\bar{u} &= 0 \\
\partial_t h\bar{u} + \partial_x \left(h\bar{u}^2 + \pi\right) + g\bar{h}\partial_x \tilde{z} &= 0 \\
\partial_t h\pi + \partial_x(h\pi\bar{u}) + c^2\partial_x\bar{u} &= \frac{1}{\epsilon}\left(\frac{gh^2}{2} - \pi\right) \\
\partial_t \tilde{z} + \bar{u}\partial_x \tilde{z} &= \frac{1}{\epsilon}(z - \tilde{z})
\end{aligned}$$

- Eigenvalues (associated to LD fields)

$$\lambda_\pm = u \pm \frac{c}{h}, \quad \lambda_u = u \ (\text{double})$$

# Suliciu relaxation solver : With source

- Solution of the Riemann problem
  Same structure as in the conservative case...
  But different intermediate states

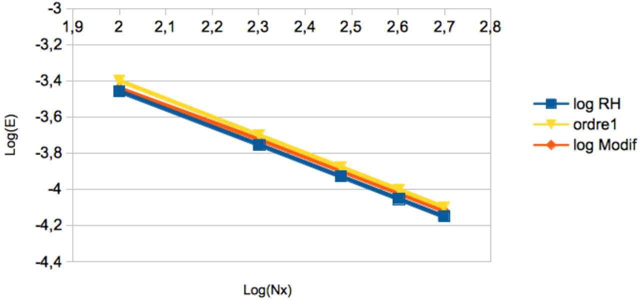



- Properties : Consistency, WB, Positivity, Entropy inequality

$$\bar{h} = h_M$$

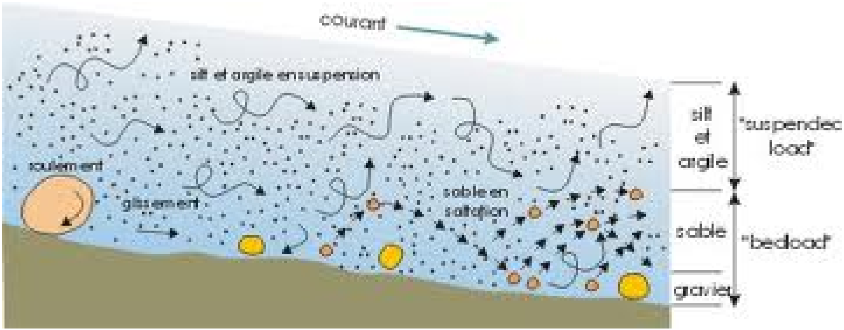
(Introduced as a wb simple Riemann solver : Galice [02])

# Suliciu relaxation solver : Numerical test




Comparison between
Suliciu without source + Hydrostatic reconstruction
and
Suliciu with source

# Suliciu relaxation solver : Towards morphodynamics




Morphodynamics process
Suspended- and bedload

# Suliciu relaxation solver : Towards morphodynamics

- Saint-Venant – Exner model

$$\partial_t h + \partial_x q_w = 0,$$

$$\partial_t q_w + \partial_x \left( \frac{q_w^2}{h} + \frac{g}{2} h^2 \right) = -gh\partial_x b - \frac{\tau_b}{\rho_w},$$

$$\rho_s (1-p)\partial_t b + \partial_x q_s = 0,$$

- Properties
  - Hyperbolic for classical choices of $q_s(h, \bar{u}, \partial_x b)$
  - Eigenvalues hard to compute except for special choices of $q_s$
  - No dynamics in the solid phase
  - No transport in the fluid phase

(Hudson [05], Castro [08], Benkhaldoun [09], Cordier [11]...)

# Suliciu relaxation solver : Towards morphodynamics

- Sediment flux formula (Grass [81])

$$q_s = A_g |\bar{u}|^2 \bar{u}$$

- Shields parameter

$$\theta = \frac{|\tau_b|/\rho_w}{g(s-1)d_m}, \qquad s = \rho_s/\rho_w$$

- Threshold sediment flux formulae

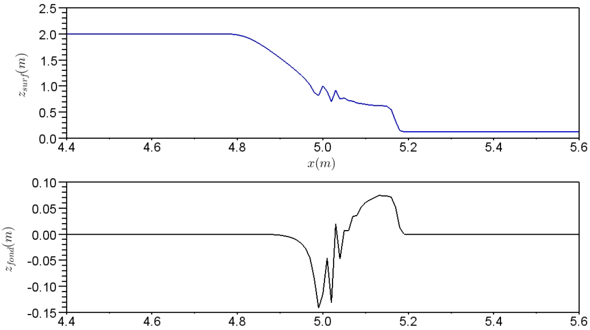
$$q_s = \Phi \sqrt{g(s-1)d_m^3}$$

  - Meyer-Peter & Müller [48] : $\Phi = 8\,(\theta - \theta_c)_+^{3/2}$
  - Engelund & Fredsoe [76]: $\Phi = 18.74(\theta - \theta_c)_+\,(\theta^{1/2} - 0.7\theta_c^{1/2})$
  - Nielsen [92] : $\Phi = 12\,\theta^{1/2}\,(\theta - \theta_c)_+ \ldots$

# Numerical strategies for bedload transport

- Steady state strategy
  - Hydrodynamic computation on fixed topography
    $\rightsquigarrow$ Steady state
  - Evolution of topography forced by hydrodynamic steady state
  - Efficient for phenomena involving very different time scales
- External coupling (time splitting)
  - Use of two different softwares for hydro- and morphodynamics
  - Allow to use existing solvers and different numerical strategies
  - Actual strategy at EDF (MASCARET-COURLIS)
  - Efficient for low coupling
- Internal coupling
  - Solution of the whole system at once
  - Need for a new solver
  - Efficient for general coupling

# Dam break : Failure for external coupling




External coupling (M. Jodeau)

# Suliciu relaxation solver : Towards morphodynamics

- Relaxation model

$$\begin{aligned}
\partial_t h + \partial_x h\bar{u} &= 0 \\
\partial_t(h\bar{u}) + \partial_x \left(h\bar{u}^2 + \pi\right) + gh\partial_x b &= 0 \\
\partial_t \pi + \bar{u}\partial_x \pi + \frac{\alpha^2}{h}\partial_x u &= \frac{1}{\epsilon}\left(\frac{gh^2}{2} - \pi\right) \\
\partial_t b + \partial_x \omega &= 0 \\
\partial_t \omega + \left(\frac{\beta^2}{h^2} - \bar{u}^2\right)\partial_x b + 2\bar{u}\partial_x \omega &= \frac{1}{\epsilon}(q_s - \omega)
\end{aligned}$$

- Definitions
  - $(\pi, \omega)$ : Auxiliary variables (fluid pressure, sediment flux)
  - $\epsilon > 0$ : (Small) relaxation parameter
  - $(\alpha, \beta) > 0$ : Have to be fixed to ensure stability

# Suliciu relaxation solver : Towards morphodynamics

- General properties
  - Formally tends to SW-Exner model when $\epsilon$ tends to 0
  - Always hyperbolic ($h \neq 0$) and LD fields
- Stability requirement

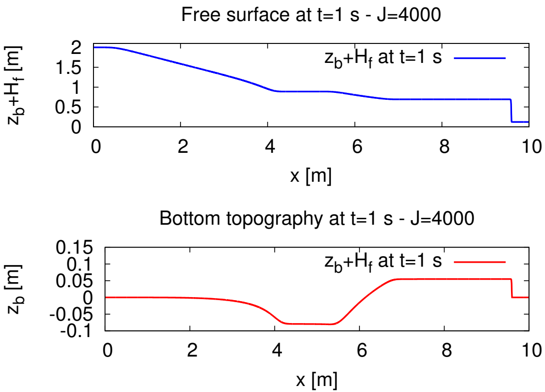
$$a^2 \geq h^2 p'(h) = gh^3, \qquad b^2 \geq (hu)^2 + gh^2 \partial_u Q_s$$

- Specificities for morphodynamics
  - No explicit dependency on sediment flux $q_S$
  - Ordered eigenvalues

$$u - \frac{b}{H} < u - \frac{a}{H} < u < u + \frac{a}{H} < u + \frac{b}{H}$$

  - Riemann problem easy to solve for $a < b$...
    But case $b < a$ leads to unafordable computations...
    With linearized version : Riemann pb may have no solution !

# Dam break : Internal coupling

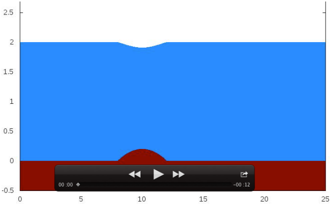



Internal coupling (O. Delestre)

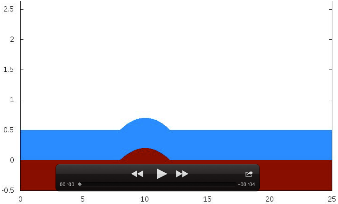
# "Steady" flow over a movable bump


Dunes


Antidunes

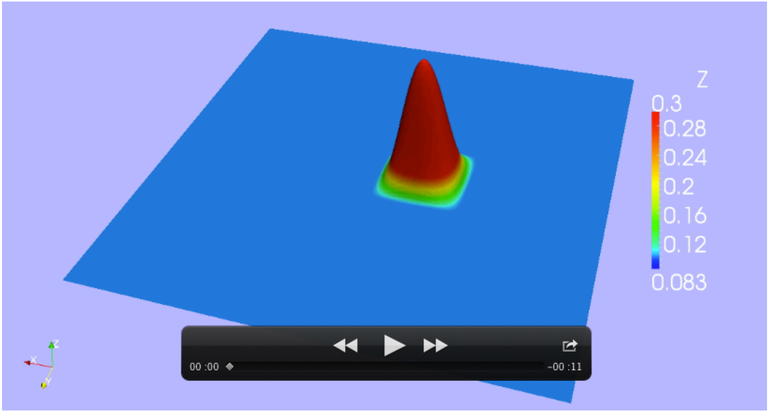



Fluvial flow




Torrential flow

# "Steady" flow over a movable bump in 2d



# Fluid models for geophysic flows

- Incompressible Navier Stokes equations
  - Efficient model for a large class of flows
  - CPU time (3d computations)
  - Complex implementation
    (moving boundaries, not connected domains...)
  - Robustness of the softwares for stiff flows
    (dam break, hydraulic jumps...)
- Shallow water equations
  - Easy implementation (fixed domain)
  - CPU time (2d computations)
  - Robustness of the software
  - Agreement with experiments... but not for all flows !
  - Constant velocity along the z-direction
  - Hydrostatic approximation

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# From hydrostatic NS equations to multilayer models

- General Idea
  - To derive a 2d (of hyperbolic type) model that takes into account vertical effects in the flow
- Derivation
  - Vertical discretization of the fluid into $M$ layers
  - Vertical integration of hydrostatic incomp. NS equ. by layer
- Consequences
  - Each layer has its own velocity
  - Coupling between the layers through
    - Pressure term (global coupling)
    - Mass exchange
      ("local" coupling : interface condition for mass equation)
    - Viscous effect
      (local coupling : interface condition for momentum equation)

# Multilayer model : Simplified case

- Hydrostatic incompressible Euler equ. with free surface

$$\begin{aligned} \partial_x u + \partial_z w &= 0, \\ \partial_t u + \partial_x u^2 + \partial_z uw + \partial_x p &= 0, \\ \partial_z p &= -g, \end{aligned}$$

with

$$t > 0, \quad x \in \mathbb{R}, \quad z_b(x) \leq z \leq \eta(t,x),$$

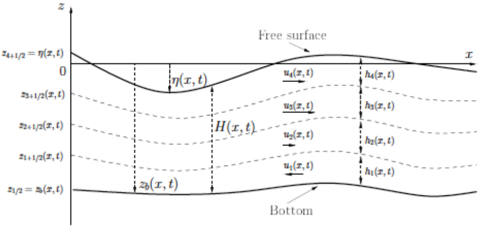
- Boundary conditions

$$\partial_t \eta + u_\eta \partial_x \eta - w_\eta = 0$$

# Multilayer model : Vertical discretization

- Velocity

$$u(t,x,z) = \sum_{\alpha=1}^{M} 1_{[z_{\alpha-1/2},z_{\alpha+1/2}]}\bar{u}_\alpha(t,x)$$



# Multilayer model : Vertical integration

- Equations after integration on a layer

$$\partial_t h_\alpha + \partial_x h_\alpha \bar{u}_\alpha = G_{\alpha+1/2} - G_{\alpha-1/2},$$
$$\partial_t h_\alpha \bar{u}_\alpha + \partial_x h_\alpha \overline{u_\alpha^2} + g h_\alpha \partial_x H$$
$$= -g h_\alpha \partial_x b + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2}$$

- Mass exchange term

$$G_{\alpha+1/2} = \partial_t z_{\alpha+1/2} + u_{\alpha+1/2} \partial_x z_{\alpha+1/2} - w_{\alpha+1/2}$$

- Natural conditions for lowest and uppest layers

$$G_{1/2} = G_{N+1/2} = 0$$

- What choice for the others ?

# Multilayer model : Variable density

- Equations after integration on a layer

$$\partial_t \rho_\alpha h_\alpha + \partial_x \rho_\alpha h_\alpha \bar{u}_\alpha = G_{\alpha+1/2} - G_{\alpha-1/2},$$

$$\partial_t \rho_\alpha h_\alpha \bar{u}_\alpha + \partial x \rho_\alpha h_\alpha \bar{u_\alpha^2} + \partial_x h_\alpha p_\alpha$$
$$= p_{\alpha+1/2} \partial_x z_{\alpha+1/2} - p_{\alpha-1/2} \partial_x z_{\alpha-1/2}$$
$$+ u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2}$$

- Mass exchange term

$$G_{\alpha+1/2} = \rho_{\alpha+1/2} \left( \partial_t z_{\alpha+1/2} + u_{\alpha+1/2} \partial_x z_{\alpha+1/2} - w_{\alpha+1/2} \right)$$

- Pressure terms

$$p_\alpha = \frac{1}{h_\alpha} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} p(x,z,t) dz, \qquad p_{\alpha+1/2} = p(x, z_{\alpha+1/2}, t),$$

# Multilayer model : Different models

- No exchange : Multifluid problems

$$G_{\alpha+1/2} = 0$$

$\rightsquigarrow$ $M$ superposed Saint-Venant systems (Castro et al. [01]...)

- Horizontal layers

$$\partial_t z_{\alpha+1/2} = \partial_x z_{\alpha+1/2} = 0 \quad \Rightarrow \quad G_{\alpha+1/2} = -w_{\alpha+1/2}$$

$\rightsquigarrow$ 1 equ. for $h_M$ and $M$ velocity equations (Rambaud [12])

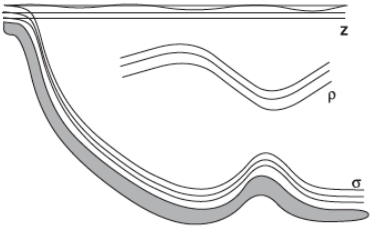
- Adaptative layers

$$h_\alpha(t,x) = \lambda_\alpha H(t,x)$$

$\rightsquigarrow$ 1 equ. for $H$ and $M$ velocity equations (ABPSa [11])

# Multilayer model : Links with classical NS discretization

- Isopycnal coordinates
  ⇝ Stratifications are imposed in the model
- $z-$ coordinates
  ⇝ Pbs to deal with complex bottom and water surface
- $\sigma-$ coordinates
  ⇝ Well-adapted to geometry but may diffuse stratifications



# Multilayer model : System of equations

$$\partial_t \sum_{\alpha=1}^{N} \rho_\alpha l_\alpha H + \partial_x \sum_{\alpha=1}^{N} \rho_\alpha l_\alpha H u_\alpha = 0$$

$$\partial_t(\rho_\alpha h_\alpha u_\alpha) + \partial_x \left(\rho_\alpha h_\alpha u_\alpha^2 + h_\alpha p_\alpha\right) = u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2} + p_{\alpha+1/2}\partial_x z_{\alpha+1/2} - p_{\alpha-1/2}\partial_x z_{\alpha-1/2}$$

$$\partial_t(\rho_\alpha h_\alpha T_\alpha) + \partial_x \left(\rho_\alpha h_\alpha T_\alpha u_\alpha\right) = T_{\alpha+1/2} G_{\alpha+1/2} - T_{\alpha-1/2} G_{\alpha-1/2}$$

$$\rho_\alpha = \rho(T_\alpha)$$

$$p_\alpha = \frac{1}{h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} p(x,z,t)dz, \qquad p_{\alpha+1/2} = p(x, z_{\alpha+1/2}, t)$$

$$G_{\alpha+1/2} = \sum^{\alpha} \frac{\partial \rho_j h_j}{\partial t} + \sum^{\alpha} \frac{\partial \rho_j h_j u_j}{\partial x}$$

# Multilayer model : Properties

- Invariant domain
  Positivity of total water depth
- Global energy inequality

$$E_{sv,\alpha}^N = \frac{\rho_\alpha h_\alpha u_\alpha^2}{2} + \frac{g\rho_\alpha (z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2)}{2}$$

$$\frac{\partial}{\partial t}\left(\sum_{\alpha=1}^N E_{sv,\alpha}^N\right) + \frac{\partial}{\partial x}\left(\sum_{\alpha=1}^N u_\alpha \left(E_{sv,\alpha}^N + \frac{g}{2}\rho_\alpha h_\alpha H\right)\right) = 0$$

- Hyperbolicity
  - Bi-layer model is hyperbolic
  - Possible local loss of hyperbolicity for more than three layers

# Numerics : Kinetic interpretation of SW system

- Saint-Venant system

$$\begin{aligned} \partial_t h + \partial_x hu &= 0, \\ \partial_t hu + \partial_x \left(hu^2 + gh^2/2\right) &= -gh\,\partial_x b \end{aligned}$$

- Density function

$$M(t,x,\xi) = \frac{h(t,x)}{c(t,x)}\chi\left(\frac{\xi - u(t,x)}{c(t,x)}\right), \qquad c = \sqrt{\frac{gh}{2}}$$

with $\chi$ an even probability function, $\xi$ a kinetic velocity

- Kinetic equation

$$\partial_t M + \xi\partial_x M - g\partial_x z \partial_\xi M = Q(x,t,\xi)$$

$\rightsquigarrow$ Equivalency with SW system : Integration in $\xi$ vs. $(1,\xi)^T$

# Numerics : Kinetic scheme (derivation, no source)

- Kinetic level
  - Definition of $M_i^n$ from $(h_i^n, u_i^n)$
  - Upwind scheme for the linear transport kinetic equation

$$f_i^{n+1} = M_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1/2}^n - f_{i-1/2}^n\right), \quad f_{i+1/2}^n = \begin{cases} \xi M_i^n & \text{if} \quad \xi > 0 \\ \xi M_{i+1}^n & \text{if} \quad \xi < 0 \end{cases}$$

- Macroscopic scheme

  Integration in $\xi$

$$(F^h)_{i+1/2}^n = \int f_{i+1/2}^n d\xi, \qquad (F^q)_{i+1/2}^n = \int \xi f_{i+1/2}^n d\xi$$

- Source term

  Perthame-Simeoni [Calcolo, 01] : Numerical integration processes

# Numerics : Kinetic scheme (properties, no source)

- Simplicity
  Macroscopic fluxes can be computed analytically if probability function $\chi$ is simple enough

$$\chi(\omega) = \frac{1}{2\sqrt{3}}\mathbb{I}_{[-\sqrt{3},\sqrt{3}]}, \qquad \chi(\omega) = \left(1 - \frac{\omega^2}{4}\right)_+^{1/2}$$

  No need for the computation of the eigenvalues
- Stability
  - Positivity for water depth
  - Entropy inequality
  - CFL condition

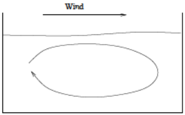
$$\Delta t^n \leq \min_i \left( \frac{\Delta x_i}{|u_i^n| + \frac{\omega_m}{\sqrt{2}}\sqrt{g h_i^n}} \right)$$

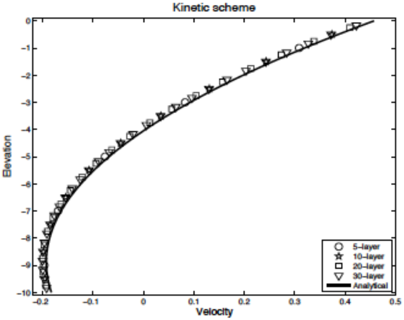
# Numerics : Multilayer system

- Extension of the kinetic interpretation
  $\rightsquigarrow$ Construction of a kinetic scheme
- Modified CFL condition
  $\rightsquigarrow$ Takes into account mass exchange terms
  $\rightsquigarrow$ Positivity of water depth and tracer concentration
- Extension of the hydrostatic reconstruction
  $\rightsquigarrow$ Preservation of relevant equilibria
- Variable density case
  - Computation of solution : Non linear problem to solve
  - Boussinesq assumption : Back to simple case

# Numerics : 3d dam break

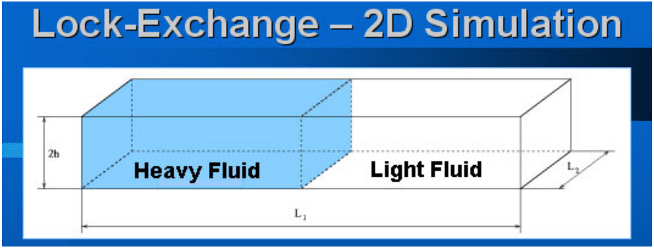
# Numerics : Wind-induced circulation



# Numerics : Stratified fluid ?

# Numerics : Lock exchange



# Numerics : Gibraltar straight

# Numerics : Ocean in bottles

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