

Modeling and numerics for (shallow) (water) flows Part 1 : Saint-Venant system

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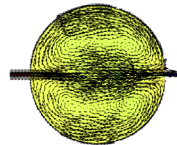
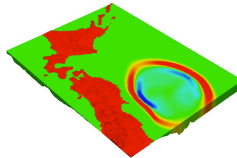
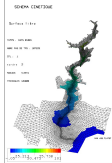
April 2, 2013

Shallow water system : Applications for EGRIN

▶ (Natural ?) hazards



▶ Simulations



Shallow water system : Equations

► 1d shallow water equations

$$\begin{aligned}\partial_t h + \partial_x hu &= 0, \\ \partial_t hu + \partial_x (hu^2 + gh^2/2) &= -gh \partial_x b + S_f\end{aligned}$$

with

h : water depth, b : bottom topography,

u : velocity of the water column, S_f : friction term



Saint-Venant (1871),

"Théorie du mouvement non-permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit", Comptes Rendus Acad. Sciences.

Shallow water system : Equations

► 1d shallow water equations

$$\begin{aligned}\partial_t h + \partial_x hu &= 0, \\ \partial_t u + u\partial_x u + g\partial_x \eta &= S'_f\end{aligned}$$

with

h : water depth, η : free surface,

u : velocity of the water column, S'_f : friction term



Saint-Venant (1871),

"Théorie du mouvement non-permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit", Comptes Rendus Acad. Sciences.

Shallow water system : Derivation (SV1)

► Wave propagation : Lagrange formula

très-petite. La première de ces intumescences s'y propage horizontalement, d'après une formule connue de Lagrange, avec une *célérité*

$$(1) \quad k = \sqrt{gh},$$

si, g étant l'accélération due à la gravité, h désigne la hauteur d'eau dans le canal à mer basse, et si nous appelons *célérité de propagation* la longueur k qu'acquiert dans l'unité de temps l'intumescence produite; acquisition que l'on ne doit point confondre avec une *vitesse* de transport réel et horizontal de l'eau; car l'intumescence ne s'allonge que parce que les molécules de la surface se soulèvent successivement, et dans des directions presque verti-

► Computation of the solution : Riemann invariant

$$k dy + U y = (U + dU)(y + dy),$$

ou

$$(9) \quad y dU = (k - U) dy;$$

ce qu'on aurait pu poser directement en égalant la quantité d'eau introduite due à la seule augmentation dU de la vitesse U , à la partie du volume de l'intumescence qui est indépendante de cette vitesse.

» Comme (2) $k = \sqrt{gy} + U$, cette équation différentielle revient à

$$(10) \quad y dU = dy \sqrt{gy}.$$

» En l'intégrant de manière que h soit la hauteur y pour $U = 0$, c'est-à-dire avant la marée ou la crue, on a

$$(11) \quad U = 2\sqrt{gh} - 2\sqrt{gh},$$

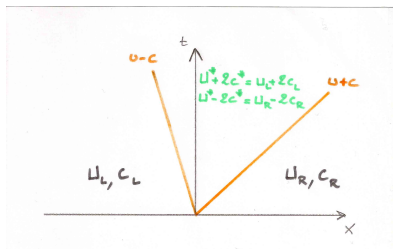
Shallow water system : Riemann invariants

- ▶ Continuous solution for homogeneous system

$$\partial_t(u + 2c) + (u + c) \partial_x(u + 2c) = 0,$$

$$\partial_t(u - 2c) + (u - c) \partial_x(u - 2c) = 0$$

with $c = \sqrt{gh}$: wave celerity



↪ Boundary conditions

Shallow water system : Derivation (SV2)

► "Physical" derivation

6. *Théorie et équations générales du mouvement non permanent des eaux courantes.* — Cette théorie, dont le Mémoire que m'a communiqué M. Parriot m'a donné l'occasion de faire l'étude, peut être présentée d'une manière directe et générale, sans invoquer la formule de propagation des ondes ou des intumescences due à Lagrange.

Soit en effet, à l'époque du temps quelconque t , dans un canal ou une rivière :

U la vitesse moyenne des eaux à travers une section transversale ω , dont s est l'abscisse, sensiblement horizontale, comptée à partir d'un endroit quelconque de son cours.

► Mass budget

1^{re} équation. Le volume ωds d'une couche comprise entre les deux sections ayant pour abscisses

$$s \quad \text{et} \quad s + ds$$

ne doit pas être changé lorsque, après un petit temps Δt , ces abscisses sont devenues

$$s + U \Delta t \quad \text{et} \quad s + ds + \left(U + \frac{dU}{ds} ds \right) \Delta t,$$

et que son épaisseur, ainsi, est devenue

$$ds + \frac{dU}{ds} ds \Delta t;$$

or, alors, la section a pour superficie

$$\omega + \left(\frac{d\omega}{ds} + \frac{d\omega}{ds} \frac{ds}{ds} \right) \Delta t = \omega + \frac{d\omega}{ds} \Delta t + \frac{d\omega}{ds} U \Delta t.$$

Égalant le produit de cette superficie par cette épaisseur, à ωds , réduisant et divisant par $\Delta t ds$, on obtient, pour la condition de continuité ou de conservation du volume,

$$(19) \quad \frac{d\omega}{dt} + \frac{d(\omega U)}{ds} = 0.$$

Shallow water system : Derivation (SV2)

► Gravity term

» Quantité de mouvement due à la pesanteur,

$$\rho g \omega ds \frac{d(\zeta + \gamma)}{ds} \Delta t;$$

► Pressure term

» Quantité de mouvement due aux pressions sur deux faces opposées ay et $a \left(\gamma + \frac{dy}{ds} ds \right)$ de la tranche,

$$\rho g a \left[\gamma \frac{\gamma}{2} - \frac{1}{2} \left(\gamma + \frac{dy}{ds} ds \right)^2 \right] dt = -\rho g \omega ds \frac{dy}{ds} \Delta t;$$

► Friction term

» Quantité de mouvement due au frottement ou à la résistance tangentielle du fond,

$$-\rho g F \chi ds;$$

► Momentum budget

» Égalant la somme des trois premières quantités à la troisième, et divisant tout par $\rho g \omega ds \Delta t$, on a la deuxième équation

$$(20) \quad \frac{d\zeta}{ds} = \frac{1}{g} \frac{dU}{dt} + \frac{U}{g} \frac{dU}{ds} + \frac{\chi}{\omega} \frac{F}{\rho g}.$$

Shallow water system : Derivation (SV2)

► Hypothesis : Almost flat bottom

» Si nous estimons les quantités de mouvement parallèlement au fond supposé très-peu incliné, nous pourrions prendre pour les grandeurs des composantes de la vitesse de la tranche et des pressions qu'elle supporte, celles de cette vitesse et de ces pressions elles-mêmes, car elles n'en diffèrent que de quantités négligeables. On aura donc :

► Hypothesis : Constant velocity

» D'autres intumescences viennent ensuite se superposer à celle-ci. L'eau qu'elles introduisent, se mêlant à celle du canal, lui fait acquérir une certaine *vitesse* horizontale que nous supposerons, en abstrayant les frottements, sensiblement la même de la surface au fond. Si nous appelons U cette vitesse acquise quand la profondeur d'eau est devenue γ , l'intumes-

► Hypothesis : Rectangular channel

2^e *équation*. Exprimons l'équilibre dynamique, pour le temps Δt , des quantités de mouvement de la même tranche, décomposées ou estimées dans une direction unique, en nous bornant à un canal à section rectangulaire de profondeur variable γ et de largeur constante a , d'où $\omega = a\gamma$.

► Conclusion

» Je donnerai, dans un autre article, l'intégration de ces équations pour un cas étendu. »

Shallow water system : Derivation (today)

- ▶ 2d shallow water equations with sources

$$\begin{aligned} \partial_t h + \nabla \cdot (h\bar{\mathbf{u}}) &= 0, \\ \partial_t (h\bar{\mathbf{u}}) + \nabla \cdot \left(h\bar{\mathbf{u}} \otimes \bar{\mathbf{u}} + \frac{gh^2}{2} \mathbf{l} \right) &= -gh\nabla b - 2\Omega \times h\bar{\mathbf{u}} - \kappa(h, \bar{\mathbf{u}})\bar{\mathbf{u}} \end{aligned}$$

- ▶ Derivation

- ▶ Gerbeau-Perthame [01]

Asymptotic expansion + Vertical integration of NS equations

- ▶ Ferrari-Saleri [04], Marche [07], Decoene [08], Frings [12]

Topography, 2d, Coriolis, Friction, Wind, Capillarity...

- ▶ Hypothesis

- ▶ Shallow water assumption \rightsquigarrow hydrostatic hypothesis
- ▶ Scaled viscosity and friction \rightsquigarrow almost constant velocity
- ▶ Almost flat bottom and free surface

Shallow water system : Properties

- ▶ 2d shallow water equations with sources

$$\begin{aligned} \partial_t h + \nabla \cdot (h\bar{\mathbf{u}}) &= 0, \\ \partial_t (h\bar{\mathbf{u}}) + \nabla \cdot \left(h\bar{\mathbf{u}} \otimes \bar{\mathbf{u}} + \frac{gh^2}{2} \mathbf{l} \right) &= -gh\nabla b - 2\Omega \times h\bar{\mathbf{u}} - \kappa(h, \bar{\mathbf{u}})\bar{\mathbf{u}} \end{aligned}$$

- ▶ Properties

- ▶ Conservation law
- ▶ Hyperbolic system (wave propagation, weak solution)
- ▶ Positivity of water depth (invariant domain, dry zones)
- ▶ Energy (entropy) (in)equality
- ▶ Non-trivial steady states

Shallow water system : Hyperbolicity

- ▶ Shallow water system

$$\begin{aligned}\partial_t h + \partial_x(h\bar{u}) &= 0 \\ \partial_t(h\bar{u}) + \partial_x\left(h\bar{u}^2 + \frac{gh^2}{2}\right) &= -gh\partial_x b\end{aligned}$$

- ▶ Quasilinear form

$$\begin{aligned}\partial_t W + A(W)\partial_x W &= S(W) \\ W &= \begin{pmatrix} h \\ h\bar{u} \end{pmatrix}, \quad A(W) = \begin{pmatrix} 0 & 1 \\ -gh + \bar{u}^2 & 2\bar{u} \end{pmatrix}\end{aligned}$$

- ▶ Hyperbolicity :

The system is *hyperbolic* iff matrix A is diagonalizable in \mathbb{R}

Shallow water system : Hyperbolicity (no source)

- ▶ Eigenvalues of shallow water system

$$\lambda_{\pm} = \bar{u} \pm \sqrt{gh}$$

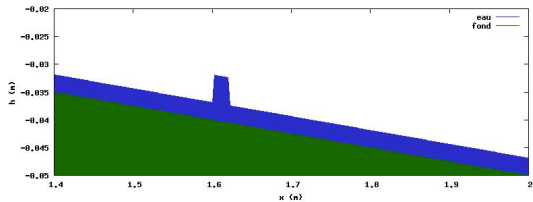
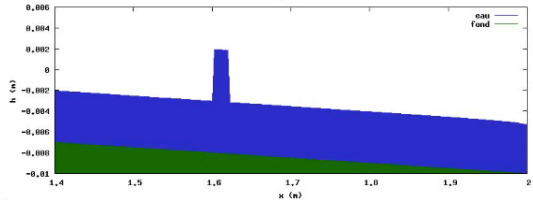
↪ SW system is strictly hyperbolic away from the vacuum
(Influence on the time/space discretization)

- ▶ Physical meaning
Introducing the Froude number

$$Fr = \frac{|\bar{u}|}{\sqrt{gh}}$$

- ▶ $Fr < 1$ ↪ **Fluvial flow** : Information goes in both directions
- ▶ $Fr > 1$ ↪ **Torrential flow** : Information only goes downstream
(Influence on the prescribed boundary conditions)

Shallow water system : Hyperbolicity



Shallow water system : Hyperbolicity (with source)

- ▶ Nonconservative form of the system

$$\begin{aligned}\partial_t h + \partial_x(h\bar{u}) &= 0 \\ \partial_t(h\bar{u}) + \partial_x\left(h\bar{u}^2 + \frac{gh^2}{2}\right) + gh\partial_x b &= 0 \\ \partial_t b &= 0\end{aligned}$$

- ▶ Eigenvalues

$$\lambda_{\pm} = \bar{u} \pm \sqrt{gh}, \quad \lambda_0 = 0$$

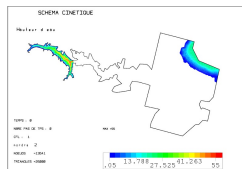
↪ Not always hyperbolic...

(Possible numerical difficulties at resonant points)

Shallow water system : Positivity / Energy

- Convex invariant domain

$$h \geq 0 \\ (h\bar{\mathbf{u}}) \in \mathbb{R}^d \quad (d = 2, 3)$$



(Possible numerical difficulties at wet/dry interfaces)

- Energy inequality

$$E = \frac{u^2}{2} + g \left(\frac{h}{2} + b \right), \quad \partial_t(hE) + \partial_x \left((\bar{u}(hE + \frac{gh^2}{2})) \right) \leq 0$$

(Can be used to ensure some stability at the discrete level)

Shallow water system : Stationary states

- ▶ Continuous cases : Momentum and Head

$$\partial_x \left(\begin{array}{c} h\bar{u} \\ \frac{\bar{u}^2}{2} + g(h+z) \end{array} \right) = 0$$

- ▶ Hydraulic jumps : RH relations

$$\left[\begin{array}{c} h\bar{u} \\ h\bar{u}^2 + \frac{gh^2}{2} \end{array} \right] = 0$$

- ▶ Two dimensional case
Much more complicate...

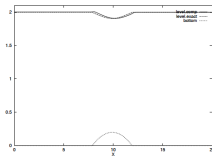
- ▶ Lake at rest

$$h + z = \eta_0, \quad \bar{\mathbf{u}} = 0$$

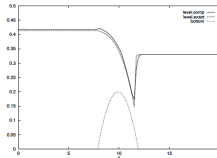
(Has to be preserved at the discrete level)

Shallow water system : Stationary states

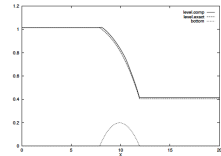
► Flow over an obstacle



Fluvial flow



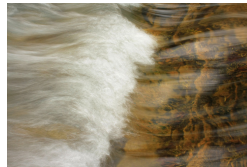
Torrential flow 1



Torrential flow 2



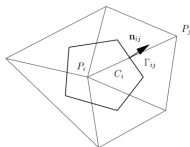
In laboratory



In river

Numerics : Finite Volume Method

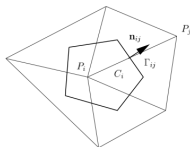
$$\partial_t u + \nabla \cdot f(u) = 0, \quad u \in \mathbb{R}^p, \quad f : \mathbb{R}^p \rightarrow \mathbb{R}^{2 \times p}$$



- ▶ Well-adapted to
 - ▶ Conservation properties
 - ▶ Discontinuous solutions
- ▶ Two main type of meshes
 - ▶ Structured grids
 - ▶ Unstructured meshes
 - ▶ Primal cells : Unknowns on the triangles
 - ▶ Dual cells : Unknowns "at the nodes"

Numerics : Finite Volume Method

$$\partial_t u + \nabla \cdot f(u) = 0, \quad u \in \mathbb{R}^p, \quad f : \mathbb{R}^p \rightarrow \mathbb{R}^{2 \times p}$$



- Integration on the prism $C_i \times [t^n, t^{n+1}]$

$$U_i^{n+1} = U_i^n - \sum_{j \in \mathcal{V}(i)} \sigma_{ij}^n F(U_i^n, U_j^n, n_{ij})$$

with

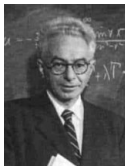
$$\sigma_{ij}^n = \frac{\Delta t^n |\Gamma_{ij}|}{|C_i|}, \quad F(U_i^n, U_j^n, n_{ij}) \approx \int_{\Gamma_{ij}} \int_{t^n}^{t^{n+1}} f(U(t, s)) \cdot n_{ij} dt ds$$

Numerics : CFL condition

► **Explicit scheme : Limitation on the time step**

Courant, R.; Friedrichs, K.; Lewy, H. (1928), "Über die partiellen Differenzengleichungen der mathematischen Physik" (in German), *Mathematische Annalen* 100 (1): 32-74 (English Version, 1967)

$$\Delta t^n \leq \frac{\Delta x}{|\max(\lambda_i^n)|}$$



R. Courant



K.O. Friedrichs



P. Lax

Numerics : CFL condition

hatten. Wir legen jetzt allgemeiner ein rechteckiges achsenparalleles Gitter zugrunde, dessen Maschenweite in der t -Richtung (Zeitmasche) gleich h und diejenige der x -Richtung (Raummasche) gleich κh mit konstantem κ ist. Das Abhängigkeitsgebiet der Differenzgleichung $u_{tt} - u_{xx} = 0$ für dieses Gitter wird ganz im Innern des Abhängigkeitsgebietes der Differentialgleichung $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$ liegen oder wird es selbst in seinem Innern enthalten, je nachdem ob $\kappa < 1$ oder $\kappa > 1$ ist.

Hieraus ergibt sich eine merkwürdige Tatsache: Läßt man im Falle $\kappa < 1$ die Maschenweite h gegen Null abnehmen, so kann die Lösung der Differenzgleichung im allgemeinen nicht gegen die Lösung der Differentialgleichung konvergieren. Ändert man nämlich etwa bei der Schwingungsgleichung (1) die Anfangswerte der Lösung der Differentialgleichung in der Umgebung der Endpunkte α und β des Abhängigkeitsgebietes (vgl. Fig. 7), so zeigt die Formel (4), daß sich auch die Lösung selbst im



Fig. 7.

Punkte (x, t) ändert. Für die Lösungen der Differenzgleichungen im Punkte S sind aber die Vorgaben in den Punkten α und β irrelevant, da diese außerhalb des Abhängigkeitsgebietes der Differenzgleichungen liegen. — Daß im Falle $\kappa > 1$ Konvergenz statthat, werden wir in § 3 beweisen. Vgl. hierzu Fig. 9, S. 62.

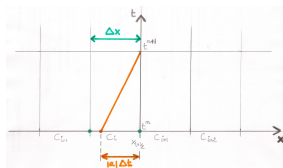
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$$\Delta t^n \leq \frac{\Delta x}{|\max(\lambda_i^n)|}$$

► Physical interpretation



$$F_{i+1/2}^n = F(U_i^n, U_{i+1}^n)$$

Information at point $x_{i+1/2}$
at time $t \in [t^n, t^{n+1}]$

has to come

from cells C_i or C_{i+1} ,

not from cells C_{i-1} or C_{i+2}

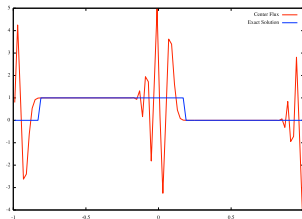
Numerics : Boundary conditions

- ▶ **Most common cases for SW flows**
 - ▶ Wall
 - ▶ Inflow/Outflow boundaries
- ▶ **Influence of the Froude number**
 - ▶ Fluvial in- or out- flow : One prescribed value
 - ▶ Torrential in (resp. out) flow : Two (resp. no) prescribed value
 - ▶ Possibly varying in time
- ▶ **Numerics**
 - ▶ Introduction of **ghost cells**
 - ▶ Neumann condition on missing data or **on Riemann invariant**
 - ▶ High order accuracy, source terms...

Bristeau-Coussin [INRIA Report 4282, 2001].

Numerics : Center flux

$$F(U, V) = \frac{f(U) + f(V)}{2}$$



Center flux for transport equation

Numerics : Properties of the solver

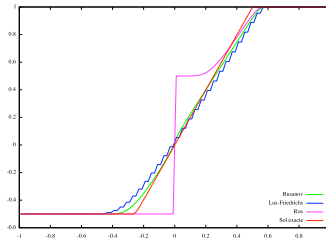
► Consistency

$$F(U, U) = f(U)$$

Necessary for convergence (Lax-Wendroff theorem)

► Non linear stability : scalar case

- Monotonicity : Convergence to the entropy solution
- TVD and L^∞ stability : Convergence up to a subsequence



Numerics : Properties of the solver

► Consistency

$$F(U, U) = f(U)$$

Necessary for convergence (Lax-Wendroff theorem)

► Non linear stability : SW system

- Convex invariant domain ($h \geq 0$)
- Discrete in-cell entropy (energy) inequality

$$E(U_i^{n+1}) - E(U_i^n) + \sigma_i^n (G(U_i^n, U_{i+1}^n) - G(U_{i-1}^n, U_i^n)) \leq 0$$

consistent with

$$\partial_t(hE) + \partial_x \left((\bar{u}(hE + \frac{gh^2}{2})) \right) \leq 0$$

Physical Requirements

Numerics : Godunov scheme

- ▶ Solve the Riemann problem

$u_{[u_l, u_r]}^R(t, x)$ solution of the system with initial data

$$u(0, x) = \begin{cases} u_l & \text{if } x \leq 0 \\ u_r & \text{if } x > 0 \end{cases}$$

Much more easy to solve than IBVProblem !

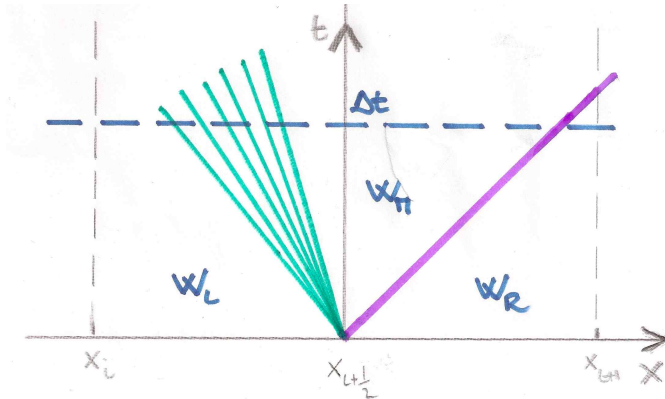
Not so easy for complex systems...

- ▶ Project the solution onto constant states

$$U_i^{n+1} = \int_{C_i^-} u_{[U_{i-1}^n, U_i^n]}^R(\Delta t^n, x) dx + \int_{C_i^+} u_{[U_i^n, U_{i+1}^n]}^R(\Delta t^n, x) dx$$

Godunov, S. K. (1959), "A Difference Scheme for Numerical Solution of Discontinuous Solution of Hydrodynamic Equations", Math. Sbornik, 47, 271–306 (English version, 1969)

Numerics : Godunov scheme



Solution of a Riemann problem for the Saint-Venant system

Numerics : Godunov scheme

Теперь мы перейдем к изучению системы уравнений

$$\frac{du}{dt} = A \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial t} = B \frac{\partial u}{\partial x} \quad (1)$$

(коэффициенты A и B мы будем предполагать постоянными). Умножив второе уравнение на λ и прибавив к первому, получим:

$$\frac{\partial(u + \lambda v)}{\partial t} = \frac{A}{\lambda} \cdot \frac{\partial(\lambda v)}{\partial x} + \lambda B \frac{\partial u}{\partial x}.$$

Если мы выберем $\lambda = \sqrt{\frac{A}{B}}$, то будем иметь:

$$\frac{\partial\left(u + \sqrt{\frac{A}{B}}v\right)}{\partial t} = \sqrt{AB} \frac{\partial\left(u + \sqrt{\frac{A}{B}}v\right)}{\partial x},$$

$$\frac{\partial\left(u - \sqrt{\frac{A}{B}}v\right)}{\partial t} = -\sqrt{AB} \frac{\partial\left(u - \sqrt{\frac{A}{B}}v\right)}{\partial x}.$$

Riemann invariants for a 2×2 system

Numerics : Godunov scheme

полными на звуковых волнах. Возьмем первое из наших уравнений

$$\frac{\partial u}{\partial t} + B \frac{\partial p(v)}{\partial x} = 0.$$

Из него следует, что по любому контуру

$$\oint_C u dx - B p(v) dt = 0.$$

В качестве следствия из этого интегрального тождества нетрудно получить, что разрывы u и p должны удовлетворять условию

$$[u] dx - B [p] dt = 0.$$

На волне, идущей вправо, $\frac{dx}{dt} = \sqrt{AB}$, и мы получаем:

$$[u] \sqrt{\frac{A}{B}} - [p] = 0.$$

Rankine-Hugoniot relations for a 2×2 system

Numerics : Godunov scheme

Литература

1. J. Neumann, R. Richtmyer. A method for the numerical calculation of hydrodynamic shocks, Journ. Appl. Physics, 21, № 3 (1950), 232–237.
2. P. D. Lax, Weak solutions of nonlinear hyperbolic equations and their numerical computation, Comm. on pure and appl. Math., VII, N 1 (1954), 159–193.
3. Н. Д. Ландау и Е. М. Lifшиц, Механика сплошных сред, Москва, Гостехиздат, 1955.
4. А. Ф. Филиппов, Об устойчивости разностных уравнений, ДАН СССР, т. 100, № 6 (1955), 1045–1048.

Some references...

Numerics : Godunov scheme

- Solve the Riemann problem

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$$u(0, x) = \begin{cases} u_l & \text{if } x \leq 0 \\ u_r & \text{if } x > 0 \end{cases}$$

Much more easy to solve than IBVProblem !

Not so easy for complex systems...

- Alternative (but equivalent !) way : Definition of a flux

$$F_{i+1/2}^n = F(U_i^n, U_{i+1}^n) = \int_0^{\Delta t^n} f(u_{[U_i^n, U_{i+1}^n]}^R(t, 0)) dt$$

↪ Valid only for conservative systems

Numerics : Approximate Riemann Solvers

▶ Basic idea

- ▶ To replace the **exact** solution by an **approximate** one
- ▶ To ensure some conservativity and consistency properties
- ▶ To use this approximate sol. to construct a numerical scheme

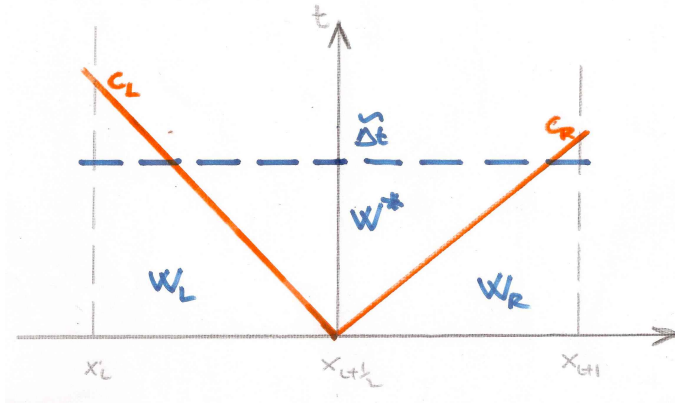
▶ Some numerical schemes

- ▶ Roe solver
- ▶ Rusanov flux
- ▶ HLL, HLLC flux
- ▶ Kinetic scheme
- ▶ Suliciu relaxation solver

▶ Some references

Godlewski-Raviart [96], Toro [99], Leveque [02], Bouchut [04]

Numerics : HLL scheme



Approximate solution for the Saint-Venant system

Numerics : HLL scheme

► Intermediate state

$$U^* = \frac{c_r U_r - c_l U_l - (f(U_r) - f(U_l))}{c_r - c_l}$$

► Numerical flux

$$F(U_l, U_r) = \begin{cases} f(U_l) & 0 < c_l \\ \frac{c_r f(U_l) - c_l f(U_r) + c_r c_l (U_r - U_l)}{c_r - c_l} & c_l < 0 < c_r \\ f(U_r) & c_r < 0 \end{cases}$$

► Source term

Source terms : Generality

- ▶ Basic definitions - with $V = (U, z)$

$$U_i^{n+1} = U_i^n - \sigma_i^n (F_l(V_i^n, V_{i+1}^n) - F_r(V_{i-1}^n, V_i^n))$$

- ▶ Consistency with the flux

$$F_l(V, V) = F_r(V, V) = F(U)$$

- ▶ Consistency with the sources

$$F_r(V_l, V_r) - F_l(V_l, V_r) = -gh(z_r - z_l) + o(z_r - z_l), \quad V_r, V_l \rightarrow V$$

- ▶ Non linear stability : Positivity & Discrete energy inequality
- ▶ Centered discretization of source terms

$$F_l(V_i^n, V_{i+1}^n) = F(U_i^n, U_{i+1}^n) + \begin{pmatrix} 0 \\ gh_i^n/2 * (z_{i+1} - z_i) \end{pmatrix}$$

Source terms : Well-balanced schemes

- ▶ Preservation of specific stationary state

$$h_i^n + z_i = \eta_0, \quad u_i^n = 0$$

- ▶ Inaccuracy of centered discretization of the sources

$$F^h(U_i^n, U_{i+1}^n) - F^h(U_{i-1}^n, U_i^n) \neq 0$$

▶ Test

- ▶ Positive and well-balanced numerical schemes
 - ▶ Extended Godunov scheme (Chinnayya-Leroux-Seguin)
 - ▶ Kinetic interpretation of source term (Perth.-Sim.,ABSM)
 - ▶ Extended Suliciu relaxation scheme v1 (Bouchut)
 - ▶ Extended Suliciu relaxation scheme v2 (Galice,ACPU)
 - ▶ Hydrostatic reconstruction (ABBKP, Liang-Marche)
 - ▶ Path-conservative scheme (Castro-Macias-Pares)
 - ▶ Hydrostatic upwind scheme (Berthon-Foucher)
 - ▶ Central scheme (Kurganov, Kurganov-Noelle...)

Source terms : Hydrostatic reconstruction

- ▶ **Problem**

Naive treatment of source terms does not preserve equilibria

↪ **Numerical oscillations**

- ▶ **Objective :**

To extend any homogeneous solver to pbs with source terms

- ▶ To preserve stationary states of the system
- ▶ To preserve stability properties of the solver

- ▶ **Key point :**

- ▶ To introduce reconstructed variables that are in equilibrium

$$F^h(\tilde{U}_i^{eq}, \tilde{U}_{i+1}^{eq}) - F^h(\tilde{U}_{i-1}^{eq}, \tilde{U}_i^{eq}) = 0$$

- ▶ To compute the deviation from this equilibrium variables
(Botta-Klein-Langenberg-Lützenkirchen [JCP, 04])

Source terms : Hydrostatic reconstruction

- ▶ Lake at rest equilibrium

$$h + z = Cst, \quad u = 0$$

- ▶ Reconstructed equilibrium water heights

$$h_{i+1/2,-}^{eq} = (h_i + z_i - z_{i+1/2})_+, \quad h_{i+1/2,+}^{eq} = (h_{i+1} + z_{i+1} - z_{i+1/2})_+$$

- ▶ Computation of the fluxes

$$F_l(V_i^n, V_{i+1}^n) = F(U_{i+1/2,-}^{eq}, U_{i+1/2,+}^{eq}) + \begin{pmatrix} 0 \\ g \frac{h_{i+1/2,-}^{eq} + h_i}{2} (z_{i+1/2} - z_i) \end{pmatrix}$$

Source terms : Hydrostatic reconstruction

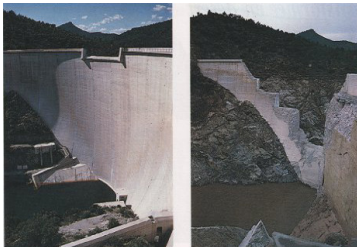
- ▶ **Consistency** :
The scheme is consistent with the equations
- ▶ **Stationary states** :
The scheme preserves the discrete lake at rest equilibrium
- ▶ **Stability properties** :

$$z_{i+1/2} = \max(z_i, z_{i+1})$$

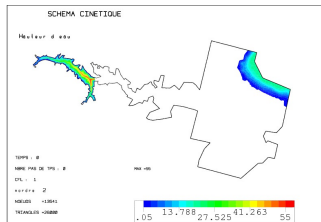
- ▶ If the homogeneous solver is positive,
the adapted solver is positive
- ▶ If the homogeneous solver satisfies a discrete entropy
inequality,
the adapted scheme satisfies a semi-discrete entropy inequality

ABBKP [JCP, 05]

Applications : Dam break in Malpasset

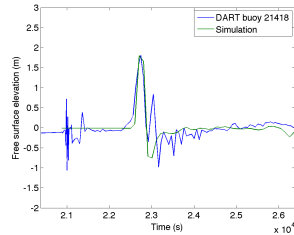
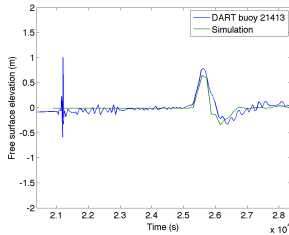


Before / After



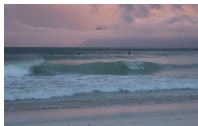
Initial data and topography

Application : Tsunami

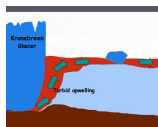
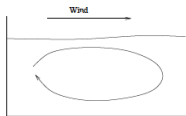


Actual problems

- ▶ **Non hydrostatic SW models** : Boussinesq type models (Dutykh-Diaz [07], Bonneton et al. [11], Sainte-Marie [11]...)



- ▶ **Hydrostatic NS equations** : Multilayer models (ABPS [11], ABPSM [11], Rambaud [12]...)



Actual problems

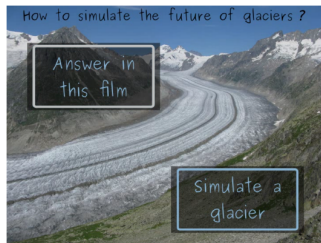
- ▶ **Shallow fluids** : Generalized topography, Alternative rheology (Savage-Hutter [91], Mangeney et al. [03], Bouchut-Westdickenberg [04], Nieto et al. [10]...)



- ▶ **Coupling phenomena** : Erosion processes (Hudson et al. [05], Castro et al. [08], Benkhaldoun et al. [09,]Cordier et al. [11], ABCDGGJSS [11]...)



Applications : Evolution d'un glacier



Author : Guillaume Jovet (FU Berlin)

mpe2013.org/fr

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"Un jour, une brève"

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