A bilayer model to study the evolution of a thin pollutant layer over water

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Outline

1 Introduction

2 Derivation of the model

3 Local energy

4 Numerical results

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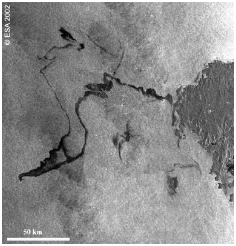
Motivation

Study some natural disasters

Oil spills in the ocean.



Motivation

Study some natural disasters

Oil spills in the ocean.

Some considerations

- Fluids with different physical properties.
- Flows with different behavior.
- Different thickness and velocity.

Goal

Deduction of a bilayer model taking into account these differences.

Approach

- Upper layer: Crude oil.
  - *Viscous fluid*
  - *Thin film slow flow* $\Rightarrow$ Reynolds lubrication equation.
- Lower layer: Water. $\Rightarrow$ shallow-water equation.
- Different thickness and velocity: $\Rightarrow$ Multiscale analysis in space and time.

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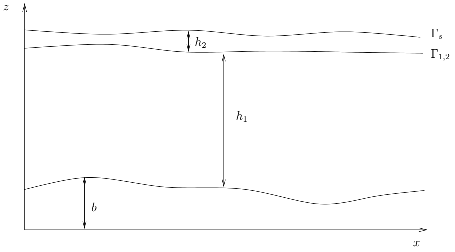
Governing equations

2D incompressible Navier-Stokes for both layers:

$$\left\{\begin{array}{l} \rho_i(\partial_t u_i + u_i\partial_x u_i + w_i\partial_z u_i) \\ \qquad\qquad = -\partial_x p_i + 2\rho_i\nu_i\partial_x(\partial_x u_i) + \rho_i\nu_i\partial_z^2 u_i + \rho_i\nu_i\partial_z(\partial_x w_i); \\ \rho_i(\partial_t w_i + u_i\partial_x w_i + w_i\partial_z w_i) \\ \qquad\qquad = -\partial_z p_i - \rho_i g + \rho_i\nu_i\partial_x^2 w_i + \rho_i\nu_i\partial_z(\partial_x u_i) + 2\rho_i\nu_i\partial_z^2 w_i; \\ \partial_x u_i + \partial_z w_i = 0; \end{array}\right. \qquad i = 1, 2.$$



- $v_i = (u_i, w_i)$: velocities.
- p_i: pressures.
- ρ_i: densities.
- ν_i: kinematic viscosities.
- $\sigma_i = 2\rho_i\nu_i D(v_i) - p_i Id$: stress tensors. $(D(v_i) = \frac{\nabla v_i + \nabla^t v_i}{2})$

Governing equations

Boundary conditions:

- Free surface ($\eta = b + h_1 + h_2$):

  - Kinematic condition: $\partial_t \eta + u_2 \partial_x \eta = w_2$;

  - Normal stress balance: $(\sigma_2 \cdot N_s)_n = -\delta \, \kappa \, N_s$.

Governing equations

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  - Kinematic condition: $\partial_t \eta + u_2 \partial_x \eta = w_2$;

  - Normal stress balance: $(\sigma_2 \cdot N_s)_n = -\delta\, \kappa\, N_s$.

- Interface ($\mathcal{I} = b + h_1$):

  - Kinematic conditions: $\partial_t h_1 + u_i \partial_x \mathcal{I} = w_i$, $i = 1, 2$.

  - Normal stress balance: $(\sigma_1 \cdot N_{\mathcal{I}})_n - (\sigma_2 \cdot N_{\mathcal{I}})_n = (\delta_{\mathcal{I}} \kappa_{\mathcal{I}} n_{\mathcal{I}})_n$.

  - Friction condition: $(\sigma_i \cdot N_{\mathcal{I}})_\tau = -c\rho_2 (v_1 - v_2)_\tau$.

Governing equations

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- Free surface ($\eta = b + h_1 + h_2$):

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- Bottom:

  - No-penetration: $v_1 \cdot N_b = 0$.

  - Friction condition: $(\sigma_1 \cdot N_b)_\tau = \alpha (v_1)_\tau$.

Derivation of the model

Steps:

1. Shallow domain assumption for each layer:

$$\epsilon = \frac{H}{L} << 1; \quad \epsilon_2 = \frac{H_2}{L} << 1.$$

2. Dimensionless / Multiscale assumption.
3. Asymptotic analysis.
4. Vertical integration and hydrostatic approximation.

Derivation of the model

Steps:

2 Dimensionless:

- Layer 1 (shallow-water)

$$x = Lx^*, \qquad z_1 = Hz_1^*, \qquad h_1 = Hh_1^*$$

$$u_1 = Uu_1^*, \qquad w_1 = \varepsilon Uw_1^*, \qquad t = Tt_1^*$$

$$p_1 = \rho_1 U^2 p_1^*, \quad Fr_1 = \frac{U}{\sqrt{gH}}; \quad Re_1 = \frac{UL}{\nu_1};$$

- Layer 2 (Reynolds)

$$x = Lx^*, \qquad z_2 = H_2 z_2^*, \qquad h_2 = H_2 h_2^*$$

$$u_2 = U_2 u_2^*, \qquad w_2 = \varepsilon_2 U_2 w_2^*, \qquad t = T_2 t_2^*$$

$$p_2 = \frac{\rho_2 \nu_2 U_2}{\epsilon_2 H_2} p_2^*, \quad Fr_2 = \frac{U_2}{\sqrt{gH_2}}; \quad Re_2 = \frac{U_2 L}{\nu_2}.$$

Derivation of the model

Steps:

2. Dimensionless:
   - Layer 1 (shallow-water)

$$x = Lx^*, \qquad z_1 = Hz_1^*, \qquad h_1 = Hh_1^*$$

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Related aspects: $\boxed{H_2 = \epsilon H}$, $\boxed{U_2 = \epsilon^2 U}$ $\rightarrow \epsilon_2 = \epsilon^2$

Derivation of the model

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   - Layer 1 (shallow-water)

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$$x = Lx^*, \qquad z_2 = \epsilon H z_2^*, \qquad h_2 = \epsilon H h_2^*$$

$$u_2 = \epsilon^2 U u_2^*, \qquad w_2 = \varepsilon^4 U w_2^*, \qquad t = \frac{1}{\epsilon^2} T t_2^*$$

$$p_2 = \frac{\rho_2 \nu_2 U}{\epsilon H} p_2^*, \qquad Fr_2 = \frac{\epsilon^2 U}{\sqrt{g\epsilon H}}; \qquad Re_2 = \frac{\epsilon U H}{\nu_2}.$$

Derivation of the model

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   - Layer 1 (shallow-water)

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$$p_2 = \frac{\rho_2 \nu_2 U}{\epsilon H} p_2^*, \quad Fr_2 = \frac{\epsilon^2 U}{\sqrt{g\epsilon H}}; \quad Re_2 = \frac{\epsilon U H}{\nu_2}.$$

$$b = Hb^*, \quad \alpha = U\alpha^*; \quad c = Uc^*; \quad C = \frac{\epsilon^2 U \rho_2 \nu_2}{\delta}, \qquad C_{\mathcal{I}} = \frac{U\rho_1\nu_1}{\delta_{\mathcal{I}}}.$$

Derivation of the model

Steps:

2. Dimensionless:
   - Layer 1 (shallow-water)

$$x = Lx^*, \qquad z_1 = Hz_1^*, \qquad h_1 = Hh_1^*$$

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$$p_2 = \frac{\rho_2 \nu_2 U}{\epsilon H} p_2^*, \qquad Fr_2 = \frac{\epsilon^2 U}{\sqrt{g\epsilon H}}; \qquad Re_2 = \frac{\epsilon U H}{\nu_2}.$$

$$z_1^* \in [b^*, \mathcal{I}] = [b^*, b^* + h_1^*]; \qquad z_2^* \in [\mathcal{I}_\epsilon, \eta_\epsilon] = [\tfrac{1}{\epsilon}(b^* + h_1^*), \tfrac{1}{\epsilon}(b^* + h_1^*) + h_2^*]$$

Derivation of the model

Steps:

3. Asymptotic analysis:

$$\begin{array}{lll} \nu_1 = \mathcal{O}(\epsilon), & \alpha = \mathcal{O}(\epsilon), & \delta_{\mathcal{I}} = \mathcal{O}(\epsilon^{-2}), \\ \nu_2 = \mathcal{O}(\epsilon), & c = \mathcal{O}(\epsilon), & \delta = \mathcal{O}(\epsilon^{-2}). \end{array}$$

$$Re_1 = \mathcal{O}(\frac{1}{\epsilon}); \quad Re_2 = \mathcal{O}(1); \quad C_{\mathcal{I}}^{-1} = \mathcal{O}(\epsilon^{-2}); \quad C^{-1} = \mathcal{O}(\epsilon^{-5})$$

and we develop the unknowns up to order 1:

$$\tilde{h}_i = h_i^0 + \varepsilon h_i^1, \quad \tilde{u}_i = u_i^0 + \varepsilon u_i^1, \quad \tilde{p}_i = p_i^0 + \varepsilon p_i^1, \quad (i = 1, 2).$$

Steps:

4 Vertical integration and hydrostatic approximation.

- Shallow water layer.
  ⋆ u_1 does not depend on z at first order.

$$\partial_t h_1 + \partial_x(h_1 u_1) = 0,$$

$$\rho_1 \partial_t(h_1 u_1) + \rho_1 \partial_x(h_1 u_1^2 + \frac{1}{2}\frac{1}{Fr^2} h_1^2) - 4\epsilon\mu_{01}\partial_x(h_1 \partial_x u_1) + \rho_1 \frac{1}{Fr^2} h_1 \partial_x b$$

$$+\frac{\rho_2}{Re_2} h_1 \partial_x(p_{2|\mathcal{I}_\epsilon}) - h_1 \mu_{01} C_{\mathcal{I}0}^{-1} \partial_x^3(b + h_1) + \rho_2 c_0 u_{1|\mathcal{I}} + \alpha_0 \gamma(h_1) u_{1|b} = 0.$$

$$\gamma(h_1) = \left(1 + \frac{\epsilon\alpha_0}{3\rho_1\mu_{01}} h_1\right)^{-1}; \qquad \epsilon\mu_{01} = \frac{1}{Re_1}$$

Derivation of the model

Steps:

4. Vertical integration and hydrostatic approximation.

   - Reynolds layer.

$$\begin{aligned} -\partial_z^2 \tilde{u}_2 + \partial_x \tilde{p}_2 &= 0 \\ \partial_z \tilde{p}_2 &= -\epsilon \beta_0; \qquad \beta_0 = \epsilon^3 \frac{Re_2}{Fr_2^2} = \mathcal{O}(1) \end{aligned}$$

Derivation of the model

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Derivation of the model

Steps:

4. Vertical integration and hydrostatic approximation.

- Reynolds layer.

$$
\begin{aligned}
-\partial_z^2 \tilde{u}_2 + \partial_x \tilde{p}_2 &= 0 \rightsquigarrow \tilde{u}_2 \\
\partial_z \tilde{p}_2 &= -\epsilon \beta_0 \rightsquigarrow \tilde{p}_2
\end{aligned}
$$

From the incompressibility equation:

$$
\partial_t \tilde{h}_2 + \partial_x \left(\int_{\mathcal{I}_\epsilon}^{\eta_\epsilon} \tilde{u}_2 dz \right) = 0.
$$

Derivation of the model

Steps:

4. Vertical integration and hydrostatic approximation.

$$\partial_{t_2}\tilde{h}_2 + \partial_x\left(\frac{1}{\epsilon^2}\tilde{h}_2\tilde{u}_1\right) + \partial_x\left(\tilde{h}_2^2\left(-\frac{1}{\epsilon c_0 Re_2} - \frac{1}{3}\tilde{h}_2\right)\partial_x\tilde{p}_2\right) = 0,$$

$$\partial_x\tilde{p}_2 = \epsilon\left(C_0^{-1}\partial_x^3\eta_\epsilon + \beta_0\partial_x\eta_\epsilon\right)$$

Steps:

4. Vertical integration and hydrostatic approximation.

- Completion of layer 1: Values $\partial_x(\tilde{p}_{2|\mathcal{I}_\epsilon})$ and $u_{1|\mathcal{I}}$

From $\tilde{p}_2$:

$$\partial_x(\tilde{p}_{2|\mathcal{I}_\epsilon}) = \epsilon \left(C_0^{-1} \partial_x^3 \eta_\epsilon + \beta_0 \partial_x (\eta_\epsilon - \mathcal{I}_\epsilon) \right)$$

Thanks to the friction condition at the interface:

$$u_{1|\mathcal{I}} = \frac{\epsilon h_2}{c_0 Re_2} \partial_x p_2$$

Final model:

$$
(\mathrm{M}_1) \equiv \begin{cases}
\partial_t h_1 + \partial_x(h_1 u_1) = 0, \\
\partial_t(h_1 u_1) + \partial_x(h_1 {u_1}^2) + \dfrac{1}{2} g \partial_x h_1^2 + g h_1 \partial_x b - 4\nu_1 \partial_x(h_1 \partial_x u_1) \\
\qquad + h_1 \left(\dfrac{\delta}{\rho_1} \partial_x^3 (b + h_1 + h_2) + r g \partial_x h_2 \right) - h_1 \dfrac{\delta_{\mathcal{I}}}{\rho_1} \partial_x^3 (b + h_1) \\
\qquad + h_2 \left(\dfrac{\delta}{\rho_1} \partial_x^3 (b + h_1) + r g \partial_x (b + h_1) \right) + \dfrac{\alpha}{\rho_1} \gamma(h_1) u_1 = 0. \\
\partial_t h_2 + \partial_x(h_2 u_1) + \partial_x \left(-{h_2}^2 \dfrac{1}{\rho_2} \left(\dfrac{1}{c} + \dfrac{1}{3\nu_2} h_2 \right) \partial_x p_2 \right) = 0.
\end{cases}
$$

with

$$
\partial_x p_2 = \delta\, \partial_x^3 (b + h_1 + h_2) + \rho_2 g \partial_x (b + h_1 + h_2), \; r = \frac{\rho_2}{\rho_1} \quad \text{and} \quad \gamma(h_1) = \left(1 + \frac{\alpha}{3\nu_1} h_1 \right)^{-1}.
$$

Normal stress balance   Friction terms

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Energy balance for M_1

Multiply:

- momentum equation for water layer by u_1
- equation of thin film flow by $\delta\partial_x^2(b + h_1 + h_2) + \rho_2 g(b + h_1 + h_2)$

and sum up the resulting equations.

Energy balance for M_1

$$
\begin{aligned}
&\partial_t\left(\frac{u_1^2}{2} + gh_1(b+\frac{h_1}{2}) + rgh_2\left(b+h_1+\frac{h_2}{2}\right)\right)\\
&-\frac{\delta}{\rho_1}\ \partial_t\left((h_1+h_2)\partial_x^2(b+h_1+h_2) + \frac{1}{2}(\partial_x(h_1+h_2))^2\right)\\
&+\frac{\delta_{\mathcal{I}}}{\rho_1}\ \partial_t\left(h_1\partial_x^2(b+h_1) + \frac{1}{2}(\partial_x h_1)^2\right)\\
&+\partial_x\left(h_1u_1\left(\frac{u_1^2}{2} + g(h_1+b)\right) + rgh_2u_1(b+2h_1+h_2)\right)\\
&-\partial_x\left(4\nu_1h_1\partial_x\left(\frac{u_1^2}{2}\right) + h_2^2\frac{1}{\rho_2}\left(\frac{1}{c} + \frac{1}{3\nu_2}h_2\right)\partial_x\left(\frac{1}{2}(\delta\partial_x^2(b+h_1+h_2) + \rho_2 g(b+h_1+h_2))^2\right)\right)\\
&-\frac{\delta}{\rho_1}\ \partial_x\Big((h_1+h_2)u_1\partial_x^2(b+h_1+h_2) - (h_1+h_2)\partial_t(\partial_x(h_1+h_2))\Big)\\
&+\frac{\delta_{\mathcal{I}}}{\rho_1}\ \partial_x\Big(h_1u_1\partial_x^2(b+h_1) - h_1\partial_t(\partial_x h_1)\Big)
\end{aligned}
$$

$$
\leq R = R_2 + \frac{\delta}{\rho_1}u_1h_2\partial_x^3\left(h_2\right) + rgu_1\partial_x\left(h_2^2\right),
$$

$$
R_2 = -4\nu_1h_1(\partial_x u_1)^2 - rg^2h_2^2\left(\frac{1}{c} + \frac{1}{3\nu_2}h_2\right)\Big(\partial_x(b+h_1+h_2)\Big)^2 - \frac{\alpha}{\rho_1}\gamma(h_1)u_1^2.
$$

Proposed modified model

$$(\mathbf{M}_2) \equiv \begin{cases} \partial_t h_1 + \partial_x (h_1 u_1) = 0, \\ \partial_t (h_1 u_1) + \partial_x (h_1 {u_1}^2) + \frac{1}{2} g \partial_x h_1^2 + g h_1 \partial_x b - 4\nu_1 \partial_x (h_1 \partial_x u_1) \\ \qquad +h_1 \left(\frac{\delta}{\rho_1} \partial_x^3 (b + h_1 + h_2) + rg\partial_x h_2 \right) - h_1 \frac{\delta_{\mathcal{I}}}{\rho_1} \partial_x^3 (b + h_1) \\ \qquad +h_2 \left(\frac{\delta}{\rho_1} \partial_x^3 (b + h_1 + h_2) + rg\partial_x (b + h_1 + h_2) \right) + \frac{\alpha}{\rho_1} \gamma(h_1) u_1 = 0, \\ \partial_t h_2 + \partial_x (h_2 u_1) + \partial_x \left(-{h_2}^2 \frac{1}{\rho_2} \left(\frac{1}{c} + \frac{1}{3\nu_2} h_2 \right) \partial_x p_2 \right) = 0. \end{cases}$$

$$\partial_x p_2 = \delta\, \partial_x^3 (b + h_1 + h_2) + \rho_2 g \partial_x (b + h_1 + h_2)$$

Added terms are of order ϵ^2.

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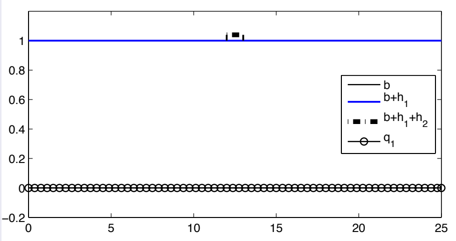
Numerical results:

Test 1: Comparison with a viscous bilayer shallow water model

$\Omega = [0, 25]$; $b(x) = 0$. $c = 1$; $\bar{\alpha} = 10^{-3}$.

- Initial conditions: $q_1(t = 0) = 0$,

$$h_1(t = 0) = 1; \quad h_2(t = 0) = \begin{cases} 0.04, & \text{if } x \in [12, 13], \\ 0, & \text{otherwise.} \end{cases}$$



- Fluids properties: Sea water $\rightarrow \rho_1 = 1027$, $\nu_1 = 10^{-6}$
  Marine residual fuel $\rightarrow \rho_2 = 920$, $\nu_2 = 5.9783 \times 10^{-4}$, $\delta = 0.033$

$$\delta_{\mathcal{I}} = \begin{cases} 0.027 & \text{if } h_2 > 0, \\ 0.072 & \text{if } h_2 = 0. \end{cases}$$

Numerical results:

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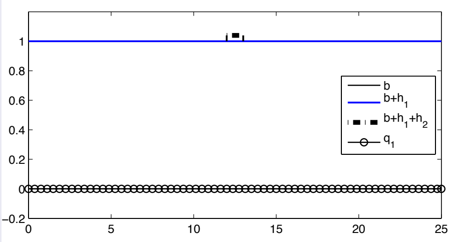

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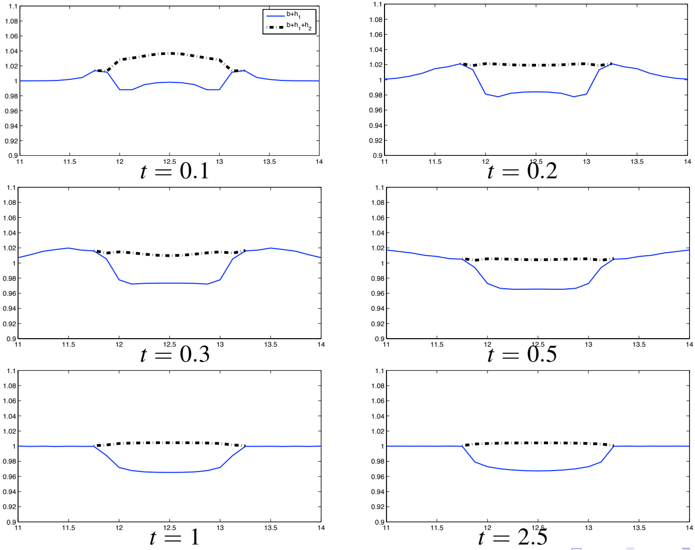
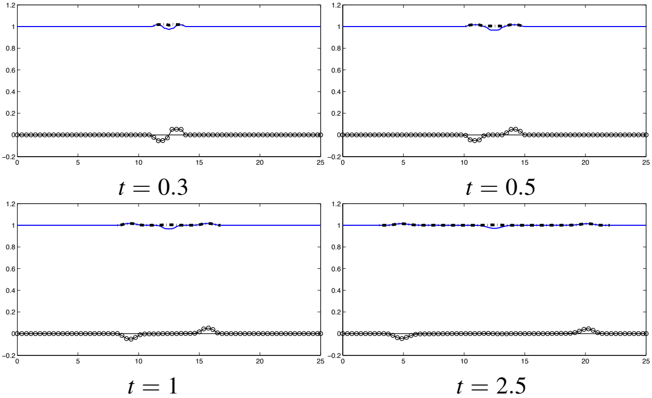
Test 1: Comparison with a viscous bilayer shallow water model

Reynolds/Saint Venant (zoom)



Test 1: Comparison with a viscous bilayer shallow water model

Saint Venant/Saint Venant



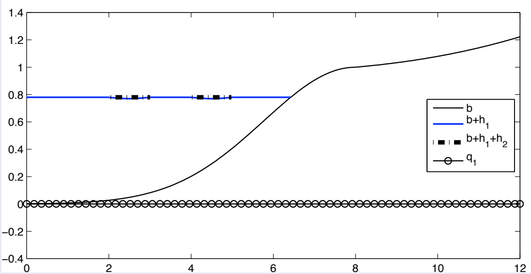
G. NARBONA-REINA, J. D. D. ZABSONRÉ, E.D. FERNÁNDEZ-NIETO, D. BRESCH. *CMES 43(1), 27-71*, 2009.

Numerical results:

Test 2: Pollutant dispersion near the coast

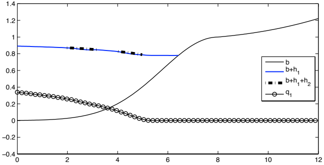
$$\Omega = [0, 12];\ b(x) = \begin{cases} \mathrm{e}^{\frac{-(x-8)^2}{10}} & \text{if } x \leq 8, \\ 1 + \mathrm{e}^{\frac{-(x-20)^2}{50}} - e^{\frac{-12^2}{50}} & \text{if } x > 8; \end{cases} \quad c = 10^{-1}$$

- Initial conditions:
  $q_1(t=0) = 0, \quad h_1(x,0) = \max(0.78 - b(x), 0) - h_2(x,0);$
  $$h_2(x,0) = \begin{cases} 2\max(\sin(2x-2), 0), & \text{if } x \in [5, 21], \\ 0, & \text{otherwise.} \end{cases}$$

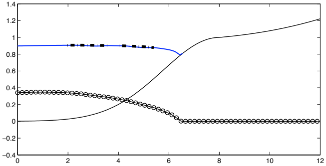


- Boundary condition: $u_1(0,t) = 0.4\sin\left(\frac{t\pi}{5}\right)$.

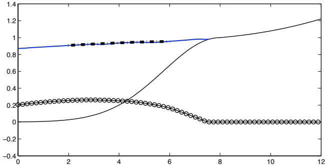
Test 2: Pollutant dispersion near the coast



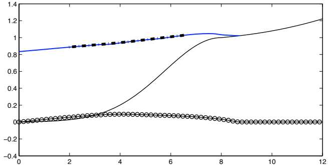
$t = 2$



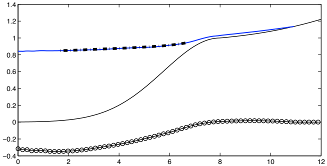
$t = 3$



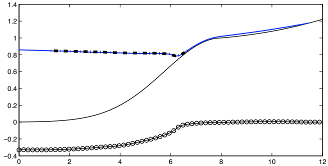
$t = 4$



$t = 5$



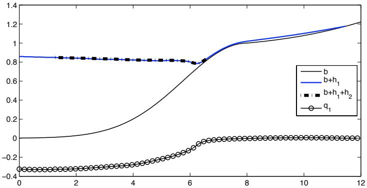
$t = 7$



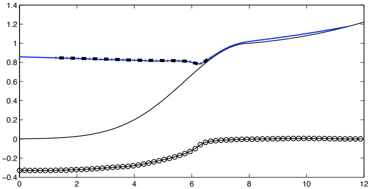
$t = 8$

Test 2: Pollutant dispersion near the coast

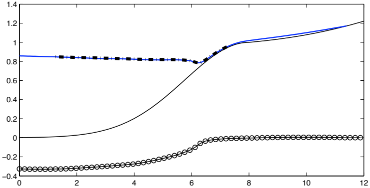
Influence of the interface friction coefficient



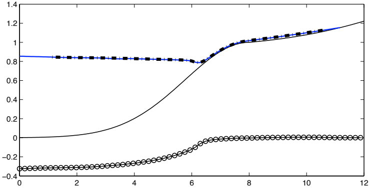
$c = 1$



$c = 10^{-1}$



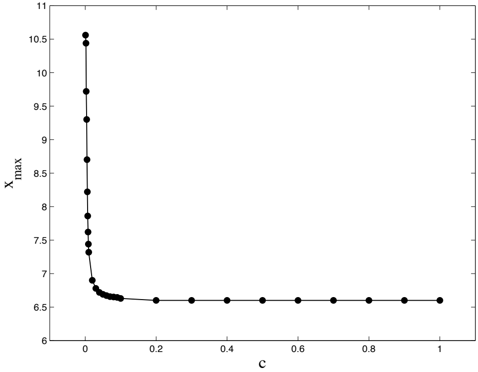
$c = 10^{-2}$



$c = 10^{-3}$

Test 2: Pollutant dispersion near the coast

Range of the pollutant spread over vs. interface friction coefficient



Conclusions and future works

Conclusions.

- We have deduced a coupled model Reynolds-Shallow Water equations.
- It is important to achieve the second order approximation to obtain viscosity and tension effects.
- The multiscale analysis in space-time is essential to modelize the pollutant transport over water.

Future works.

- Derivation of a 2d model.
- Non-Newtonian properties for the pollutant layer?

F. Gerbeau, B. Perthame. *Disc. Cont. Dynam. Syst. Series B 1(1), 89–102,* 2001.

A. Oron, S. H. Davis, S. G. Bankoff. *Rev. Mod. Phys. 69(3), 931–980,* 1997.

Je vous remercie de votre attention