A numerical method for a depth averaged Euler system in two dimensions

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Motivations

Better understand dispersive phenomena in order to better predict geophysical situations.

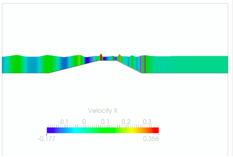
(a) Dam break

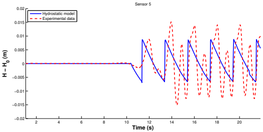
(b) Tsunami

(c) Mascaret

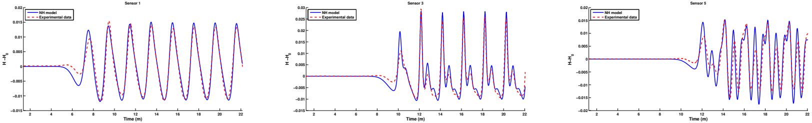
(d) Hydraulic jump

Dispersive effects: Dingemans experiment





Hydrostatic model



Non-hydrostatic model

A depth-averaged model

$$\begin{aligned}
\frac{\partial H}{\partial t} + \frac{\partial Hu}{\partial x} + \frac{\partial Hv}{\partial y} &= 0 \\
\frac{\partial Hu}{\partial t} + \frac{\partial}{\partial x}(Hu^2) + \frac{\partial}{\partial y}(Huv) + \frac{\partial}{\partial x}(g\frac{H^2}{2}) + \frac{\partial}{\partial x}(Hp_{nh}) &= -(gH + 2p_{nh})\frac{\partial z_b}{\partial x} \\
\frac{\partial Hv}{\partial t} + \frac{\partial}{\partial y}(Hv^2) + \frac{\partial}{\partial x}(Huv) + \frac{\partial}{\partial y}(g\frac{H^2}{2}) + \frac{\partial}{\partial y}(Hp_{nh}) &= -(gH + 2p_{nh})\frac{\partial z_b}{\partial y} \\
\frac{\partial Hw}{\partial t} + \frac{\partial Hwu}{\partial x} + \frac{\partial Hwv}{\partial y} &= 2p_{nh} \\
-\frac{\partial Hu}{\partial x} - \frac{\partial Hv}{\partial y} + u\frac{\partial (H+2z_b)}{\partial x} + v\frac{\partial (H+2z_b)}{\partial y} &= 2\bar{w}
\end{aligned}$$

- Average velocity
  $u(x,t) = \frac{1}{H}\int_{z_b}^{\eta} u(x,z,t)dz$,
- Green-Naghdi class model
  - M.-O.B., A. Mangeney, J.S-.M., and N. Seguin. An energy-consistent depth-averaged Euler system: derivation and properties. *Discrete Contin. Dyn. Syst. Ser. B*, 20(4):961–988, 2015.
  - J.S.-M. Vertically averaged models for the free surface Euler system. Derivation and kinetic interpretation. *Math. Models Methods Appl. Sci. (M3AS)*, 21(3):459–490, 2011.

- Energy balance
  $\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(u(E + g\frac{H^2}{2} + Hp_{nh})) = 0$

Dual operators shallow water version

- **Objective** : Apply a projection-correction scheme
  - Rewrite the model as Euler
  - Define the operators div_{sw} and ∇_{sw} :

$$\nabla_{sw}(f) = \begin{pmatrix} H\frac{\partial f}{\partial x} + f\frac{\partial (H+2z_b)}{\partial x} \\ H\frac{\partial f}{\partial y} + f\frac{\partial (H+2z_b)}{\partial y} \\ -2f \end{pmatrix}$$

$$\text{div}_{sw}(\mathbf{v}) = \frac{\partial H v_1}{\partial x} + \frac{\partial H v_2}{\partial y} - v_1\frac{\partial (H+2z_b)}{\partial x} v_2\frac{\partial (H+2z_b)}{\partial y} + 2v_3$$

which verify:

$$\int_\Omega \text{div}_{sw}(\mathbf{v}) f dx = -\int_\Omega \nabla_{sw}(f)\cdot\mathbf{v}\, dx + \int_{\partial\Omega} H f \mathbf{v}\cdot\mathbf{n}\, d\sigma$$

  - Remark that the operators are H and z_b dependent.

Euler formulation

We can rewrite the model:

$$\begin{array}{rcl}\dfrac{\partial H}{\partial t}+\nabla_0\cdot(H\mathbf{u}) & = & 0\\ \dfrac{\partial H\mathbf{u}}{\partial t}+\nabla_0\cdot(H\mathbf{u}\otimes\mathbf{u})+\nabla_0(\dfrac{g}{2}H^2)+\nabla_{sw}(p) & = & -gH\nabla_0(z_b)\\ \mathrm{div}_{sw}(\mathbf{u}) & = & 0\end{array}$$

where we note $\mathbf{u}=\begin{pmatrix}u\\ w\end{pmatrix}$ the velocity of the Depth-averaged Euler model and

$$\nabla_0=\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ 0\end{pmatrix}$$

Fractional time scheme with incremental version

- Hyperbolic step

$$
\begin{aligned}
H_i^{n+1/2} &= H_i^n - \sum_{j\in K_i} \sigma_{ij}\mathcal{F}_H(H_i^n, H_j^n) - \sigma_i \mathcal{F}_H(H_i^n, H_{e,i}^n) \\
(H\mathbf{u})_i^{n+1/2} &= (H\mathbf{u})_i^n - \sum_{j\in K_i} \sigma_{ij}\mathcal{F}_{(H\mathbf{u})}((H\mathbf{u})_i^n, (H\mathbf{u})_j^n) \\
&\quad - \sigma_i \mathcal{F}_{(H\mathbf{u})}((H\mathbf{u})_i^n, (H\mathbf{u})_{e,i}^n)
\end{aligned}
$$

- Correction step

$$
\begin{aligned}
H_i^{n+1} &= H_i^{n+1/2} \\
(H\mathbf{u})_i^{n+1} &= (H\mathbf{u})_i^{n+1/2} - \Delta t \nabla_{sw}(p^{n+1})_i \\
\mathrm{div}_{sw}(\mathbf{u}^{n+1}) &= 0.
\end{aligned}
$$

Correction step: Variational formulation

- Correction step: solve

$$\begin{aligned} H^{n+1} &= H^{n+1/2} \\ (H\mathbf{u})^{n+1} + \Delta t \, \nabla_{sw} \, p^{n+1} &= (H\mathbf{u})^{n+1/2} \\ \mathrm{div}_{sw} \left(\mathbf{u}^{n+1}\right) &= 0 \end{aligned}$$

Variational mixed problem

Find $\mathbf{u} \in V$ and $p \in Q$ such that

$$\begin{aligned} \frac{1}{\Delta t^n} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= \frac{1}{\Delta t^n} a(\mathbf{u}^{n+1/2},\mathbf{v}), \quad \forall \mathbf{v} \in V, \\ b(\mathbf{u},q) &= 0, \quad \forall q \in Q. \end{aligned}$$

with

$$Q = \{q \in L^2(\Omega)\},$$

and

$$V = \{\mathbf{v} = (v_1, v_2) \in (L^2(\Omega))^3 | \mathrm{div}_{sw} \, (\mathbf{v}) \in L^2(\Omega)\},$$

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= \int_\Omega H\mathbf{u}\cdot\mathbf{v}\, dx, \quad \forall \, \mathbf{u},\mathbf{v} \in V, \\ b(\mathbf{v},q) &= -\int_\Omega \mathrm{div}_{sw} \, (\mathbf{v}) q \, dx, \quad \forall \mathbf{v} \in V, \forall \, q \in Q, \end{aligned}$$

Correction step: Variational formulation

- Correction step: solve

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- Pressure equation

Shallow water version of the Laplacian equation

$$\Delta_{sw}\, p = \frac{1}{\Delta t^n} \mathrm{div}_{sw}\, (\mathbf{u}^{n+1/2})$$

with

$$\Delta_{sw}\, p = \mathrm{div}_{sw}\, \left(\frac{1}{H} \nabla_{sw}\, p \right).$$

Correction step: Variational formulation

- Correction step: solve

$$\begin{aligned} H^{n+1} &= H^{n+1/2} \\ (H\mathbf{u})^{n+1} + \Delta t\, \nabla_{sw}\, p^{n+1} &= (H\mathbf{u})^{n+1/2} \\ \text{div}_{sw}\, \left(\mathbf{u}^{n+1}\right) &= 0 \end{aligned}$$

Variational mixed problem

Find $\mathbf{u} \in V$ and $p \in Q$ such that

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- Compatibility of the boundary conditions

Hyperbolic part

- Distinguish Fluvial flow / Torrential flow
- Consider a Riemann problem at the interface

Correction part

- Dirichlet or Neumann boundary condition for the pressure
- Slide boundary condition for the velocity

Numerical approximation

Discrete problem

$$\begin{pmatrix} \frac{1}{\Delta t^n} A_H & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} \frac{1}{\Delta t^n} A_H U^{n+1/2} \\ 0 \end{pmatrix},$$

$$M_H = \left(\int_\Omega H \varphi_i \varphi_j dx \right)_{1 \leq i, j \leq N}, \quad A_H = \begin{pmatrix} M_H & 0 & 0 \\ 0 & M_H & 0 \\ 0 & 0 & M_H \end{pmatrix}.$$

and the two matrices B^t, B defined by

$$B^t = \left(\int_\Omega \nabla_{sw}(\phi_l) \varphi_i dx \right)_{1 \leq l \leq M, 1 \leq i \leq N}, B = -\left(\int_\Omega \text{div}_{sw}(\varphi_j) \phi_l dx \right)_{1 \leq l \leq M, 1 \leq j \leq N}$$

2D finte volume/ Finite element method

- Choice of a stable pair of finite elements.

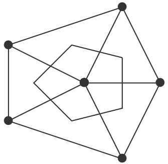


Figure : Representation of the dual mesh (finite volume mesh)

- Iterative method to solve the linear problem : Conjugate Gradient - Uzawa algorithm: $U^{(0)}, P^{(0)} given,$ while $||BU^{(k)}|| > tolerance$:

$$\begin{aligned} solve \quad A_H U^{(k+1)} &= A_H U^{n+1/2} - \Delta t B^t P^{(k)} \\ compute \quad P^{(k+1)} &= P^{(k)} + \alpha B U^{(k)}\ ,\ k > 0 \end{aligned}$$

Properties of the scheme from the 1D scheme

- **Positivity of H**

CFL condition of the kinetic scheme for the shallow water model $\Rightarrow$ Positivity of $H^{n+1/2}$
Splitting scheme $\Rightarrow H^{n+1} = H^{n+1/2}$

- **The lake at rest**

Lake at rest $\Leftrightarrow Hu = 0$ and $H + z_b = Cste$
Elliptic equation $\Rightarrow p = 0$
Satisfied by the hydrostatic part with the hydrostatic reconstruction.

- **Discrete entropy in time**
- **Inf-sup condition**

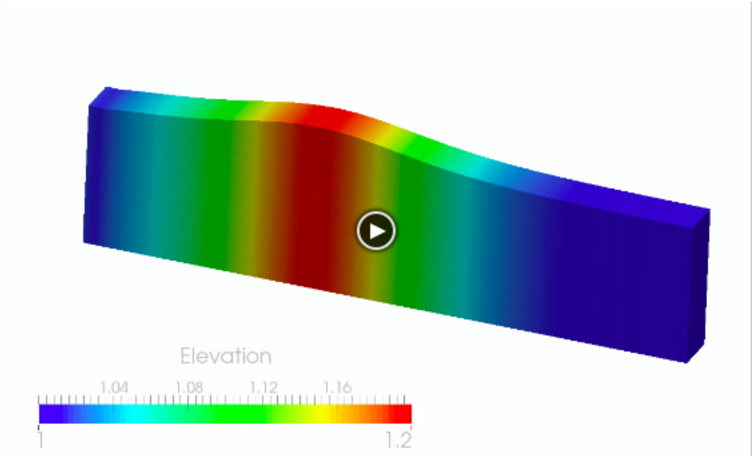
• N.A., M.-O.B., E.G., and J.S-M. A combined finite volume - finite element method for a dispersive shallow water system *Networks and Heterogeneous Media (NHM)*, pages 1-27, 2016.
• N.A., M.-O.B., E.G., and J.S-M. robust and stable numerical scheme for a depth-averaged Euler system. submitted, 2015
• N.A., M.-O.B., E.G., and J.S-M. A two dimensional method for a dispersive shallow water model. In progress.

Numerical test for an analytical solution

- Propagation of a solitary wave

$$H = H_0 + a\left(\text{sech}\left(\frac{x-ct}{l}\right)\right)^2$$

$$H_0 = 1.0 \text{ m}, a = 0.2 \text{ m}, c, l \in \mathbb{R}$$

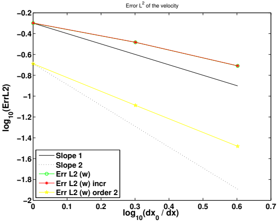


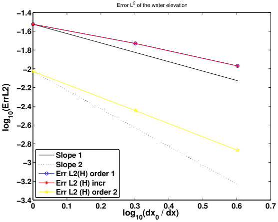
Convergence rate

- Propagation of a solitary wave

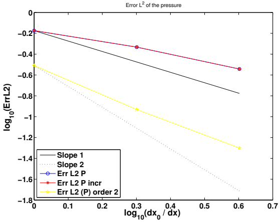
$$H = H_0 + a\left(\text{sech}\left(\frac{x-ct}{l}\right)\right)^2$$

$$H_0 = 1.0 \text{ m}\ , a = 0.3 \text{ m}\ , c, l \in \mathbb{R}$$





(a) Water depth

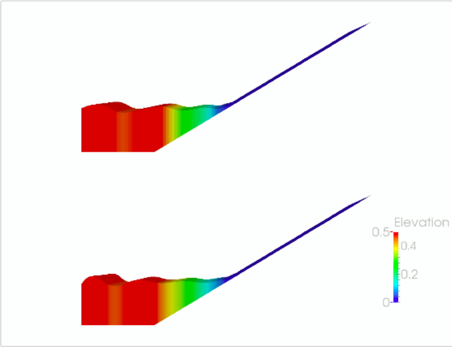


(b) Pressure

Wet/dry interface

Pressure equation

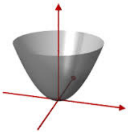
$$\Delta_{sw}\, p = \frac{1}{\Delta t^n} \mathrm{div}_{sw}\left(\mathbf{u}^{n+1/2}\right) \quad \text{with } \Delta_{sw}\, p = \mathrm{div}_{sw}\left(\frac{1}{max(H,\epsilon)} \nabla_{sw}\, p\right).$$



Thacker's solution

- Add a source term for the momentum equation:

$$\frac{\partial Hw}{\partial t} + \frac{\partial Huw}{\partial x} + \frac{\partial Hvw}{\partial y} - 2p = Hs.$$

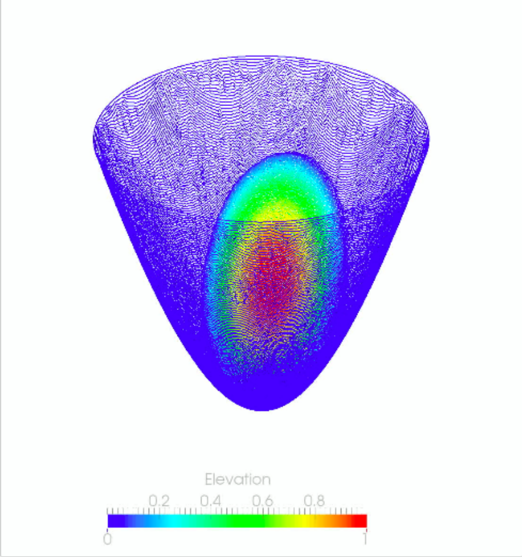


- solution :

$$\begin{aligned}
H(x,y,t) &= \max(0, H_0 - \frac{\alpha}{2}\left(x - a\cos(\sqrt{r}t)\right)^2 - \frac{\beta}{2}\left(y - a\sin(\sqrt{r}t)\right)^2,) \\
u(x,y,t) &= -a\sqrt{r}\sin(\sqrt{r}t), \\
v(x,y,t) &= a\sqrt{r}\cos(\sqrt{r}t), \\
w(x,y,t) &= -\alpha a\sqrt{r}\sin(\sqrt{r}t)x + \alpha a\sqrt{r}\cos(\sqrt{r}t)y, \\
p(x,y,t) &= \frac{a^2\alpha r}{2}H, \\
s(x,y,t) &= \alpha ar\sin(\sqrt{r}t)x - \alpha ar\cos(\sqrt{r}t)y, \\
z_b(x,y) &= \frac{\alpha}{2}(x^2 + y^2),
\end{aligned}$$

where $a, \alpha > 0$ and $a\alpha < 1, \quad r = \frac{\alpha g}{1-\alpha^2 a^2}.$

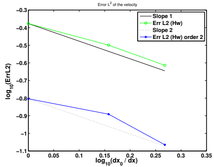
Thacker's result



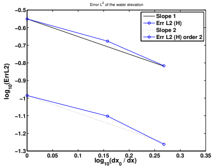
Thacker's test - Comparison with analytical solution

Convergence rate

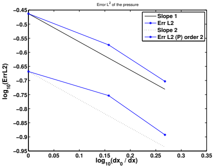
- First and second order



(c) Velocity



(d) Water depth



(e) Pressure

Conclusion and outlook

Properties

Discrete entropy
Equilibria

Validation of the scheme

Analytical solution
Comparison with experimental data 1D

Two dimensional

Variational formulation
Finite element method
Iterative method

Outlook

Higher order
Geophysical case : Tsunami ...
Optimization

Thank you !