Discrete asymptotic equations for long wave propagation

S. Bellec, M. Colin and M. Ricchiuto

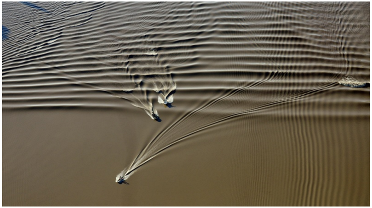
May, the 26th 2016

Context

Aim: Modeling wave transformation in near-shore zones.

- dispersive effects;
- nonlinear effects;

Sumatra 2004 tsunami reaching the coast of Thailand (from Madsen et al.2008)



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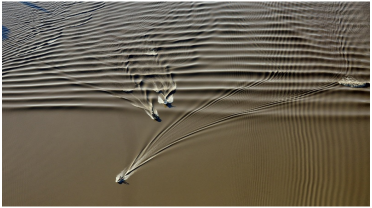

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New method

Euler Equations → Discrete Euler Equations → Discrete Asymptotic.

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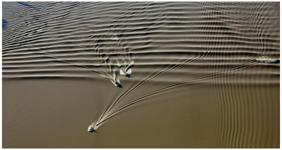

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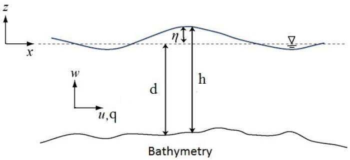
Euler system of equations

Euler system

$$\begin{cases} \partial_t u + u\partial_x u + w\partial_z u + \partial_x p = 0 \\ \partial_t w + u\partial_x w + w\partial_z w + \partial_z p + g = 0 \\ \partial_x u + \partial_z w = 0 \\ \partial_z u - \partial_x w = 0 \end{cases} \quad (1)$$

B.C. :

- in $z = \eta$: $w = \partial_t \eta + u\partial_x \eta,\ p = 0$
- in $z = -d$: $w = -u\partial_x d$



Parameters

nonlinearity parameter $\varepsilon = \frac{a}{d_0}$

dispersion parameter $\sigma = \frac{d_0}{L}$

New variables :

$$\tilde{x} = \frac{x}{L}, \ \tilde{z} = \frac{z}{d_0}, \ \tilde{t} = \frac{\sqrt{gd_0}}{L} t, \ \tilde{\eta} = \frac{\eta}{a}, \ \tilde{d} = \frac{d}{d_0},$$

$$\tilde{u} = \frac{d_0}{a\sqrt{gd_0}} u, \ \tilde{w} = \frac{L}{a} \frac{1}{\sqrt{gd_0}} w, \ \tilde{p} = \frac{p}{gd_0 \rho},$$

We express u in function of the depth-average velocity $\bar{u}$ using an asymptotic expansion.

Euler system of equations

Euler system

$$\begin{cases} \varepsilon\partial_t u + \varepsilon^2 u\partial_x u + \varepsilon^2\sigma^2 w\partial_z u + p_x = 0 \\ \varepsilon\sigma^2\partial_t w + \varepsilon^2\sigma^2 u\partial_x w + \varepsilon^2\sigma^2 w\partial_z w + p_z + 1 = 0 \\ \partial_x u + \partial_z w = 0 \\ \partial_z u - \sigma^2\partial_x w = 0 \end{cases} \quad (2)$$

B.C. :

- in $z = \varepsilon\eta$: $w = \partial_t\eta + \varepsilon u\partial_x\eta,\ p = 0$
- in $z = -d$: $w = -u\partial_x d$

Hypothesis : $\sigma^2 << 1$

If we neglect terms of order σ^2 :

Saint-Venant equations

$$\begin{cases} \eta_t + [h\bar{u}]_x = 0 \\ \bar{u}_t + \bar{u}\bar{u}_x + g\eta_x = 0 \end{cases} \tag{3}$$

Hypothesis : $\sigma^2 << 1$

If we neglect terms of order σ^4:

Green-Naghdi equations

$$\begin{cases} \eta_t + [h\bar{u}]_x = 0 \\ (1+\tau[h,d])\,\bar{u}_t + \bar{u}\bar{u}_x + g\eta_x - \dfrac{1}{h}\left[\left(\dfrac{h^3}{3} - \dfrac{h^2 d}{2}\right)\mathcal{P} + \dfrac{h^2}{2}\mathcal{Q}\right]_x \\ + d_x\Big((h-d)\mathcal{P} + \mathcal{Q}\Big) = 0 \end{cases}$$

where $\mathcal{P} = \bar{u}\bar{u}_{xx} - \bar{u}_x\bar{u}_x$, $\mathcal{Q} = \bar{u}(d\bar{u})_{xx} - \bar{u}_x(d\bar{u})_x$,
$\tau[h,d]\bar{u} = -\frac{1}{h}\left(\frac{h^3}{3}\bar{u}_x + \frac{h^2}{2}d_x\bar{u}\right)_x + d_x(h\bar{u}_x + d_x\bar{u})$.

Boussinesq System. Hypothesis : $\sigma^2 << 1$, $\varepsilon << 1$

Weakly nonlinear models $\varepsilon = \mathcal{O}(\sigma^2)$

Neglecting terms of order σ^4 and $\varepsilon\sigma^2$:

Peregrine equations

$$\begin{cases} \eta_t + [h\bar{u}]_x = 0 \\ \bar{u}_t + \bar{u}\bar{u}_x + g\eta_x + \frac{d^2}{6}\bar{u}_{txx} - \frac{d}{2}[d\bar{u}]_{txx} = 0 \end{cases} \quad (3)$$

$\rightsquigarrow$ There exists other Boussinesq type system: Beji-Nadaoka equations, Madsen-Sorensen equations, Nwogu equations...

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Notations

• The matrices $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{Q}$ are the usual mass, derivation, and stiffness matrices.

• For given columns vectors $A = (a_i)_{0\leq i\leq N}$ and $B = (b_i)_{0\leq i\leq N}$, we have introduced the operator $\diamond$:

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with
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Non-dimensionnal discrete Euler Equations

$$\varepsilon \frac{d}{dt}\mathcal{M}U + \frac{\varepsilon^2}{3}\left(\mathcal{N}(U^2) + U \diamond (\mathcal{N}U)\right) + \mathcal{N}P = \mathcal{O}(\varepsilon^2\sigma^2) \tag{5}$$

$$\varepsilon\sigma^2 \frac{d}{dt}\mathcal{M}W + \frac{d}{dz}\mathcal{M}P + \mathcal{I} = \mathcal{O}(\varepsilon^2\sigma^2) \tag{6}$$

$$\mathcal{N}U + \mathcal{M}\frac{d}{dz}W = 0, \tag{7}$$

$$\mathcal{M}\frac{d}{dz}U - \sigma^2\mathcal{N}W = 0. \tag{8}$$

The boundary conditions become

- at the free surface

$$\mathcal{M}\hat{W} = \frac{d}{dt}\mathcal{M}E + \frac{\varepsilon}{3}\Big(\mathcal{N}(E \diamond \hat{U}) - E \diamond (\mathcal{N}\hat{U}) + 2\hat{U} \diamond (\mathcal{N}E)\Big), \tag{9}$$

$$\mathcal{M}\hat{P} = 0, \tag{10}$$

- at the bottom

$$\mathcal{M}\check{W} = -\frac{1}{3}\Big(\mathcal{N}(D \diamond \check{U}) - D \diamond (\mathcal{N}\check{U}) + 2\check{U} \diamond (\mathcal{N}D)\Big). \tag{11}$$

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Asymptotic expansions

$$U(t,z) = U^0(t) + \mathcal{O}(\sigma^2), \tag{12}$$

$$W = -(z\mathcal{K}U^0 + [D; U^0]) + \mathcal{O}(\sigma^2). \tag{13}$$

It is natural to introduce the following bracket, where $\mathcal{K} = \mathcal{M}^{-1}\mathcal{N}$

$$[A; B] = A \diamond (\mathcal{K}B) + \frac{1}{3}\left(\mathcal{K}(A \diamond B) - \mathcal{M}^{-1}(A \diamond (\mathcal{N}B)) + 2\mathcal{M}^{-1}(B \diamond (\mathcal{N}A))\right)$$

$$U = U^0 - \sigma^2\left(\frac{z^2}{2}\mathcal{K}^2 U^0 + z[D; U^0]\right) + \mathcal{O}(\sigma^4). \tag{14}$$

Integrating through the depth and reversing the relation

$$U^0 = \bar{U} + \sigma^2\left(\frac{D^2}{6} \diamond (\mathcal{K}^2\bar{U}) - \frac{D}{2} \diamond (\mathcal{K}[D; \bar{U}])\right) + \mathcal{O}(\varepsilon\sigma^2, \sigma^4). \tag{15}$$

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Expansions of U and P

$$U = \bar{U} - \sigma^2 \left(\left(\frac{z^2}{2} - \frac{D^2}{6} \right) \diamond (\mathcal{K}^2 U^0) + \left(z + \frac{D}{2} \right) \diamond (\mathcal{K}[D; U^0]) \right) + \mathcal{O}(\varepsilon\sigma^2, \sigma^4). \tag{16}$$

$$P = \varepsilon E - z\mathcal{I} + \varepsilon\sigma^2 \left(\frac{z^2}{2} \mathcal{K}\bar{U} + z[D; \bar{U}] \right) + \mathcal{O}(\varepsilon^2\sigma^2, \varepsilon\sigma^4). \tag{17}$$

$$E_t + [H; \bar{U}] + B = \mathcal{O}(\varepsilon\sigma^2, \sigma^4),$$

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New numerical scheme for Peregrine equations

$$
\begin{cases}
\dfrac{d}{dt}\mathcal{M}E + \mathcal{M}[H;\bar{U}] = 0 \\[2ex]
SW + \mathcal{M}\dfrac{d}{dt}\left(\dfrac{D^2}{6}\diamond(\mathcal{K}^2\bar{U}) - \dfrac{D}{2}\diamond\mathcal{K}[D;\bar{U}]\right) = 0.
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Major differences in the schemes :

- discretization of   in the continuity equation.
- Dispersive terms in the momentum equation : $[D;\bar{U}]$ vs $\{D;\bar{U}\}$.

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Linear Peregrine equations

$$\begin{cases} \eta_t + [d\,\bar{u}]_x = 0 \\ \bar{u}_t + g\eta_x + \dfrac{d^2}{6}\bar{u}_{txx} - \dfrac{d}{2}[d\,\bar{u}]_{txx} = 0 \end{cases} \tag{19}$$

Exact solution for a constant bottom

$$\eta(t,x) = A\cos(kx - \omega t),\ \bar{u}(t,x) = \frac{\omega}{kd}A\cos(kx - \omega t)$$

where $\omega = \sqrt{\frac{gdk}{1+k^2d^2/3}}$.

Phase velocity

Numerical solution with $d = 13$, $A = 0.05$ and $k = 2\pi/15$.

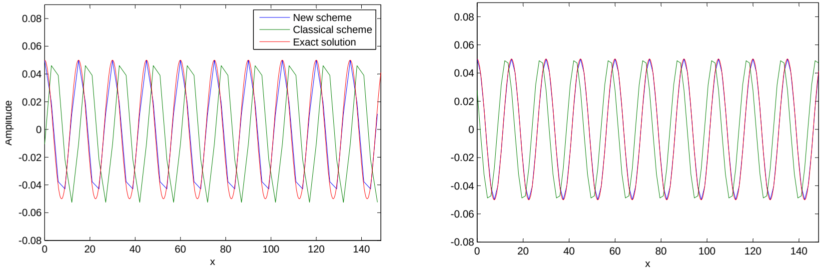


Figure: Evolution of a traveling periodic wave for the two numerical schemes: Left : $\Delta_x = 3$ ($N_\lambda = 5$). Right : $\Delta_x = 1.5$ ($N_\lambda = 10$).

Grid convergence

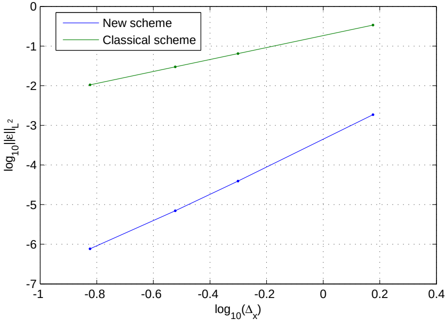
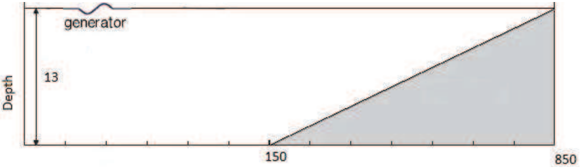
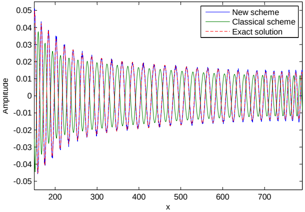


Figure: Grid convergence results for a periodic traveling wave.

linear shoaling gradient





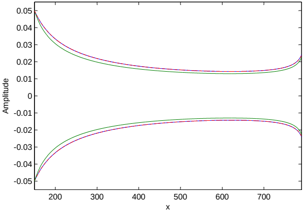


Figure: Left: Shoaling wave profiles of Peregrine schemes ($\Delta_x = 0.85$). Right: Theoretical envelope of the two numerical schemes ($\Delta_x = 0.85$).

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Conclusion and Perspectives

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- Adapt the method with other Euler discretization.
- Discretize firstly Euler in z.
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THANK YOU