Dynamic behaviour of the static/flowing interface in viscoplastic granular flows

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▷ F. Bouchut, I.R. Ionescu, A. Mangeney, *An analytic approach for the evolution of the static/flowing interface in viscoplastic granular flows*, Comm. Math. Sci., 2016.

▷ C. Lusso, F. Bouchut, A. Ern, A. Mangeney, *A simplified model for static/flowing dynamics in thin-layer flows of granular materials with yield*, 2015.

▷ C. Lusso, F. Bouchut, A. Ern, A. Mangeney, *Explicit solutions to a free boundary model for the static/flowing transition in granular flows*, 2015.

1 *Viscoplastic materials and granular rheology*

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1. Viscoplastic materials and granular rheology

Viscoplastic materials are characterized by

- Nonlinear relation between stress and strain rate
- Irreversible deformations inducing dissipation

FIG.: Snow avalanche, Mud flow

Mathematical description

▷ time t, position $x \in \Omega \subset \mathbb{R}^N$,
▷ velocity field $u(t, x) \in \mathbb{R}^N$,
▷ stress tensor $\sigma(t, x)$ symmetric $N \times N$ matrix,
▷ force field $f(t, x) \in \mathbb{R}^N$.

▷ Incompressibility

$$\text{div } u = 0,$$

▷ Momentum conservation

$$\partial_t u + u \cdot \nabla u - \text{div } \sigma = f,$$

▷ Initial condition

$$u(0, x) = u^0(x),$$

▷ Neumann boundary condition (for example)

$$\sigma \mathbf{n} = 0 \text{ on } \partial\Omega.$$

▷ The *rheology* gives a relation between the *stress tensor* σ and the *strain rate* Du defined by $(Du)_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$.
▷ A rheology with *yield stress* is characterized by

$$\sigma = -p\mathrm{I}_N + 2\nu Du + \kappa \frac{Du}{\|Du\|},$$

where $\|A\|^2 = \frac{1}{2}\sum_{i,j=1}^{N} A_{ij}^2$, p is the pressure, $\kappa \geq 0$ is the yield stress, and $\nu \geq 0$ is the viscosity.
▷ Note that $Du/\|Du\|$ is *multivalued* : when $Du = 0$, it can be any trace-free symmetric matrix with norm less or equal to one. It corresponds to *plug zones* or *static zones* (where there is a solid rotation movement).
Thus at locations (t,x) such that $Du(t,x) = 0$, $\sigma(t,x)$ just needs to satisfy $\|\sigma(t,x) + p(t,x)\mathrm{I}_N\| \leq \kappa(t,x)$.
▷ A relevant rheology for granular materials is with a *Drucker-Prager yield stress*,

$$\kappa = \mu_s p_+,$$

where p_+ is the positive part of the pressure, and $\mu_s > 0$ is the *internal friction*, a coefficient characterizing the material.

2. Static/flowing interface

▷ In the previous flow description, the *static (solid) domain* where $Du = 0$ and the *flowing (fluid) domain* where $Du \neq 0$ are not defined explicitly. They evolve according to the equations set on the whole domain.

How to describe the dynamics of the static/flowing interface (also called yield surface) ?

▷ For a mass flowing down a topography, a relevant configuration is as follows

free surface

U(Z)

interface

h

flowing phase

topography

B(X)

b

θ(X) static phase

X

x
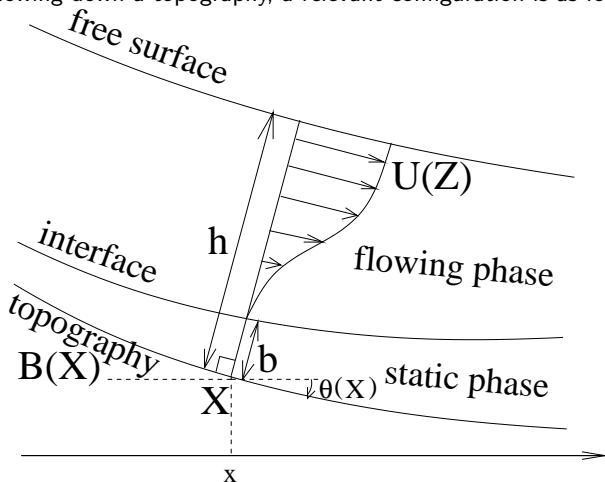
▷ Models for the dynamics of the static/flowing interface are proposed for example by Aranson & Tsimring 02, Khakhar et al. 01... A review of the existing models is given in Iverson & Ouyang 15.

▷ They are based on phenomenological equations or on strong assumptions such as specified velocity profile, or reducing the flow to a sliding block. These models make a thin-layer assumption.

▷ For example, the BCRE model states that

$$\partial_t b = \frac{g \cos \theta (\mu_s - |\tan \theta|)}{\partial_Z U},$$

where g is the gravity constant, θ is the slope angle, and U is the downslope component of the velocity. In this model the velocity is assumed to be linear in terms of the normal variable Z. Then one needs an equation for the evolution of the slope $\partial_Z U$.

3. The new model

In a series of work, we have derived and evaluated a model for the dynamics of the static/flowing interface, with the following features/assumptions

- It is derived rigorously (i.e. without heuristics) from the Drucker-Prager viscoplastic model by asymptotic expansion
- Thin-layer assumption, small topography curvature, small viscosity
- The internal friction is close to the slope $\mu_s \sim |\tan\theta|$ (gravity and friction more or less compensate, letting the possibility to have the static and flowing phases)
- The velocity is small (it is possible because of the previous assumption)
- The pressure is convex with respect to the normal variable Z (condition of stability of the double layer static/flowing configuration)
- No specific velocity profile (not a depth-average model !)
- New nonlinear non-hydrostatic pressure correction

We denote by $U(t, X, Z)$ the velocity in the downslope direction, $h(t, X)$ the width, $b(t, X)$ the interface position, and $\theta(X) < 0$ the topography slope angle.

> **Theorem [Bouchut, Ionescu, Mangeney, 2016]**
> With the previous assumptions, the solution to the viscoplastic model in the configuration between topography and free surface (with suitable boundary conditions) satisfies
> $$\partial_t \left(h - \tfrac{h^2}{2} d_X \theta \right) + \partial_X \left(\int_0^h U dZ \right) = 0,$$
> $$\partial_Z U > 0 \quad \text{for } Z > b(t, X),$$
> $$\partial_t U + S - \partial_Z(\nu \partial_Z U) = O(\epsilon^2) \quad \text{for } Z > b(t, X), \qquad (1)$$
> $$\text{with} \quad S = g(\sin \theta + \partial_X(h \cos \theta)) - \partial_Z(\mu_s p),$$
> the boundary conditions
> $$\nu \partial_Z U = 0 \text{ at } Z = h(t, X), \qquad (2)$$
> $$U = 0 \text{ at } Z = b(t, X), \qquad (3)$$
> $$\nu \partial_Z U = 0 \text{ at } Z = b(t, X), \qquad (4)$$
> the static equilibrium condition
> $$S(t, X, b(t, X)) \geq 0, \qquad (5)$$
> and the nonhydrostatic pressure
> $$p = g \left(\cos \theta + \sin \theta \partial_X h - 2| \sin \theta| \frac{\partial_X U}{|\partial_Z U|} \right) \times (h - Z) + O(\varepsilon^3), \quad \text{for } Z > b(t, X).$$

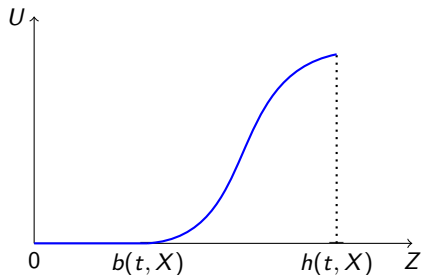For each time $t > 0$, the velocity profile looks like on the figure below.

FIG.: Typical velocity profile with respect to Z at fixed time $t > 0$, satisfying the boundary conditions. The velocity vanishes over $[0, b(t, X)]$ and increases on $(b(t, X), h(t, X))$.

Initially we can take for example a profile with a flat part for $Z < b^0$, and a linear part for $Z > b^0$. The boundary conditions are not necessarily satisfied initially.

- The problem has to be solved for $b(t, X) < Z < h(t, X)$, meaning that the static part $Z < b(t, X)$ has been eliminated.

- The pressure convexity assumption ensures that $\partial_Z S \leq 0$, and hence that the monotonicity $\partial_Z U > 0$ is preserved. It also ensures the stability of the static layer in the sense that the yield condition is satisfied without specifying what is the stress in the static region. Without the convexity assumption, some material could start flowing inside the static region, leading to unstable band shearing.

- The boundary conditions (4) $\nu \partial_Z U(t, X, b(t, X)) = 0$ expresses the continuity of the shear stress accross the interface. It is an extra condition (with respect to a problem with fixed boundary) that determines the evolution of the moving interface $b(t, X)$.

- The static equilibrium condition (5) expresses that the shear stress in the static layer must be less than the yield stress for this layer to remain static. Without this condition the whole layer would flow down. Mathematically it is like an entropy condition that selects the physical solution.

- The pressure has two nonhydrostatic contributions. The first is linear in $h - Z$, it is only due to the slope of the free surface $\partial_X h$. The second is more involved, it is nonlinear in terms of the gradient of U, proportional to $\partial_X U / \partial_Z U$. This one induces a pressure feedback effect to the velocity that depends on the inhomogeneities in the downslope direction X.

- Fully static solutions $U \equiv 0$, $b = h$, $\partial_t h = 0$ are characterized by the static equilibrium condition (5). This condition says that the slope of the free surface is at most μ_s, the friction must be higher than the effect of gravity through the slope.
- Some steady flows with static/flowing transition exist if $\nu = 0$:
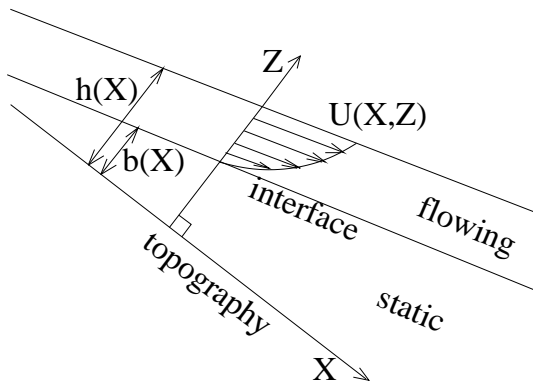
The free surface and interface have slope equal to μ_s, so that gravity exactly balances internal friction. The shape of the profile is arbitrary.

In the case without viscosity $\nu = 0$, the interface position $b(t, X)$ satisfies

$$\partial_t b(t, X) = \frac{S(t, X, b(t, X))}{\partial_Z U(t, X, b(t, X))}, \qquad \text{if } \partial_Z U(t, X, b(t, X)) \neq 0.$$

- It we take into account only the hydrostatic pressure in the value of S and neglect $\partial_X h$, we recover the BCRE equation.
- Our profile of U is arbitrary however, and $\partial_Z U$ is evaluated at $Z = b(t, X)$.
- If $\partial_Z U(t, X, b(t, X)) = 0$ this differential equation is no longer valid, indeed one has also $S(t, X, b(t, X)) = 0$, hence 0/0 above.
- If $\nu > 0$ the differential equation becomes

$$\partial_t b = \nu \left(\frac{\partial_Z S - \nu \partial^3_{ZZZ} U}{S} \right)_{Z=b}, \qquad \text{if } S(t, X, b(t, X)) \neq 0.$$

There is a third-order derivative, while the parabolic equation on U is only second-order. This makes the problem quite difficult to stabilize numerically.

▷ 2D simulations of the viscoplastic model are performed in Martin, Ionescu, Mangeney, Bouchut, Farin (2016).

▷ They show that the analytic expansion of our model with the correction in $\partial_X U / \partial_Z U$ is a better approximation to the 2D computed pressure than the hydrostatic one.

4. Simplified model

We consider that X is fixed, leading to the problem of finding $U(t, Z)$ defined for $b(t) < Z < h$ and $b(t)$, $0 < b(t) < h$ (h given) such that

$$\partial_t U + S - \partial_Z(\nu \partial_Z U) = 0 \quad \text{for } b(t) < Z < h,$$

where $S(t, Z)$ is a given source satisfying $\partial_Z S \leq 0$, with boundary conditions

$$
\begin{aligned}
U &= 0 && \text{at } Z = b(t), \\
\nu \partial_Z U &= 0 && \text{at } Z = b(t), \\
\nu \partial_Z U &= 0 && \text{at } Z = h,
\end{aligned}
$$

and the static equlilibrium condition

$$S(t, b(t)) \geq 0.$$

▷ One can prove that if $\nu > 0$, the static equlilibrium condition is automatically satisfied.

▷ The feedback process and the X dependency is hidden in the assumed knowledge of the source $S(t, Z)$.

A simple numerical method for the simplified model is as follows (Lusso, Bouchut, Ern, Mangeney, 2016).

▷ We solve the problem over $[0, h]$ with the boundary condition at $Z = h$ ($\partial_Z U = 0$) and an additional condition at $Z = 0$ ($U = 0$) by

$$\frac{U_j^{n+1/2} - U_j^n}{\Delta t^n} + S(t^n, Z_j) - \nu \frac{U_{j+1}^n + U_{j-1}^n - 2U_j^n}{\Delta Z^2} = 0,$$

(under the CFL condition $2\nu\Delta t^n \leq \Delta Z^2$).

▷ We then simply cut the negative values

$$U_j^{n+1} = \max(U_j^{n+1/2}, 0).$$

▷ The interface position b^n is then recovered as

$$b^n = (k^n - 1)\Delta Z, \qquad k^n = \min\left\{j \in \{1, \ldots, n_Z\} \text{ such that } U_j^n \geq C_0\Delta Z^2\right\},$$

where C_0 is an appropriate constant of the order of S/ν.

It means that U merely vanishes for $Z < b^n$, and $U > 0$ for $Z > b^n$.

Numerical experiments show the following dynamic behaviour when taking S constant, $S = g \cos \theta (\mu_s - |\tan \theta|)$ (hydrostatic pressure).

▷ The static/flowing interface position $b(t)$ first decreases as a consequence of viscosity until a time t^c, and attains a minimal value $b^{\min}$ (starting phase with erosion).

▷ Then $b(t)$ increases as a consequence of friction, and (if h is sufficiently large) reaches an asymptotic regime with upward velocity $\dot{b}^{\infty}$ (stopping phase with deposition), before fully stopping at attaining h.
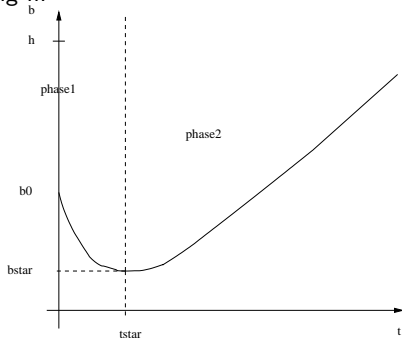
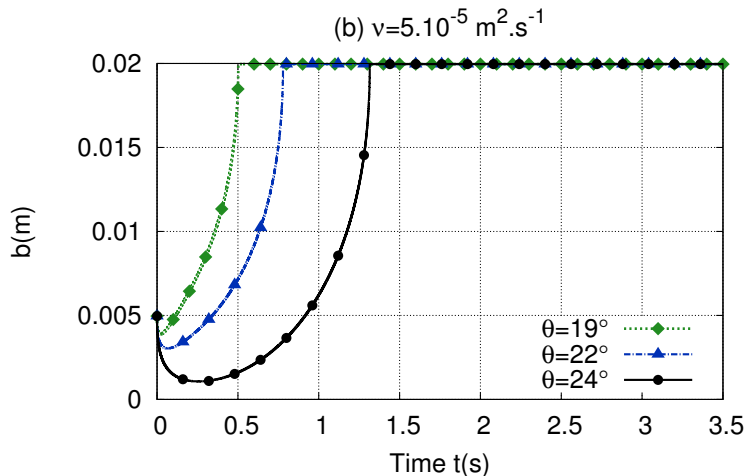FIG.: Evolution of the thickness of the static/flowing interface as a function of time.

Fig.: Static/flowing interface position b as a function of time t for different slope angles using an initially linear velocity profile with shear 70s^{-1}, and viscosity $\nu = 5 \cdot 10^{-5}\text{m}^2\text{s}^{-1}$, $\mu_s = \tan(26^o)$, $h = 2$cm, $b^0 = 5$mm.
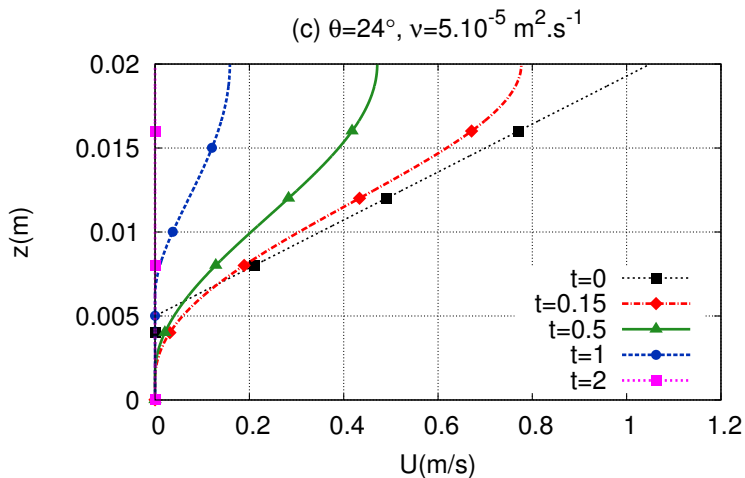
FIG.: Velocity profile $U(Z)$ at different times, with an initially linear velocity profile with shear $70s^{-1}$, for a viscosity $\nu = 5 \cdot 10^{-5} m^2 s^{-1}$ and slope angle $\theta = 24^o$, $\mu_s = \tan(26^o)$, $h = 2cm$, $b^0 = 5mm$.
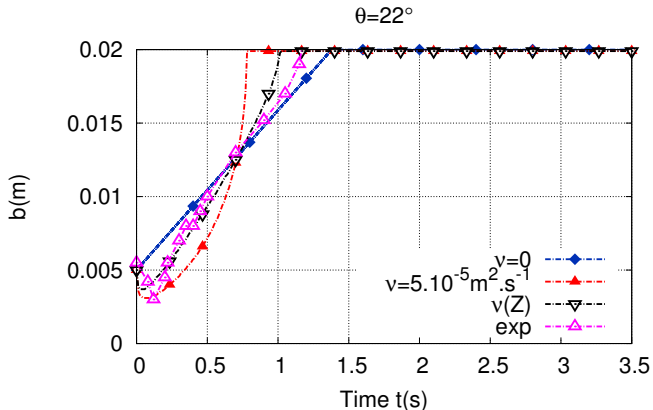
FIG.: Static/flowing interface position b as a function of time t with linear initial velocity profile with shear $70s^{-1}$ and slope angle $\theta = 22^{o}$, $\mu_s = \tan(26^{o})$, $h = 2$cm, $b^0 = 5$mm, for respectively experimental measurements from Farin, Mangeney, Roche 2014, our model without viscosity, with constant viscosity $\nu = 5 \cdot 10^{-5}$m^{2}s^{-1}, or variable viscosity associated with the $\mu(I)$ law.

FIG.: Velocity profiles $U(Z)$ at time $t = 0.5$s with linear initial velocity profile with shear 70s^{-1} and slope angle $\theta = 22^o$, $\mu_s = \tan(26^o)$, $h = 2$cm, $b^0 = 5$mm, for respectively experimental measurements from Farin, Mangeney, Roche 2014, our model without viscosity, with constant viscosity $\nu = 5 \cdot 10^{-5}\text{m}^2\text{s}^{-1}$, or variable viscosity associated with the $\mu(I)$ law.
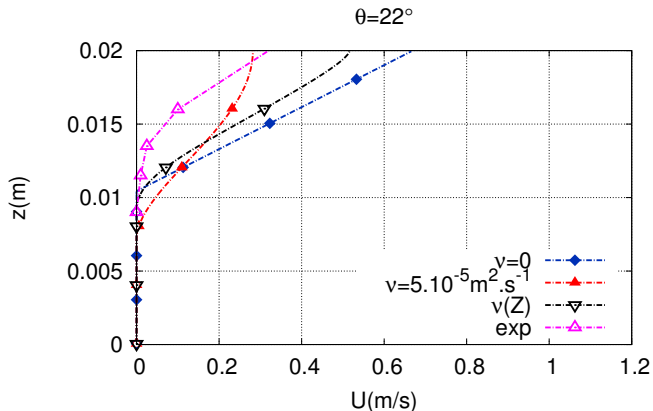
5. Dynamics with Z-dependent source term

▷ The previous study was with S constant (independent of Z), i.e. we consider only the hydrostatic part of the pressure.

▷ In the case with Z-dependent source term $S(t, Z)$, one would like to know what is the influence of this dependency on the evolution of $U(t, Z)$.

▷ Recall that in the coupled problem, S is

$$S = g(\sin\theta + \partial_X(h\cos\theta)) - \partial_Z(\mu_s p),$$

with

$$p = g\left(\cos\theta + \sin\theta\partial_X h - 2|\sin\theta|\frac{\partial_X U}{|\partial_Z U|}\right) \times (h - Z).$$

thus

$$\partial_Z S = -\mu_s\partial_{ZZ}^2 p = 2\mu_s g|\sin\theta|\partial_{ZZ}^2\left(\frac{\partial_X U}{|\partial_Z U|}(h - Z)\right)$$

relates mainly to the downslope space inhomogeneities.

With a Z-dependent source $S(t, Z)$ and assuming no viscosity $\nu = 0$, one can build the solution $U(t, Z)$ explicitly.

▷ It has to solve
$$\partial_t U + S = 0 \quad \text{for } Z > b(t) \quad \text{with } \partial_Z U > 0,$$

▷ with boundary condition
$$U = 0 \quad \text{at } Z = b(t),$$

▷ and static equlilibrium condition

$$S(t, b(t)) \geq 0.$$

It looks like an ODE in time, but it is slightly more complicate than that. We assume that

▷ $\partial_Z S \leq 0$, and S has a unique zero $b^*(t)$ satisfying $S(t, b^*(t)) = 0$. Then

$$S(t, Z) > 0 \quad \text{for all } Z < b^*(t),$$
$$S(t, Z) < 0 \quad \text{for all } Z > b^*(t).$$

▷ The static equilibrium condition then says that $b(t) \leq b^*(t)$.

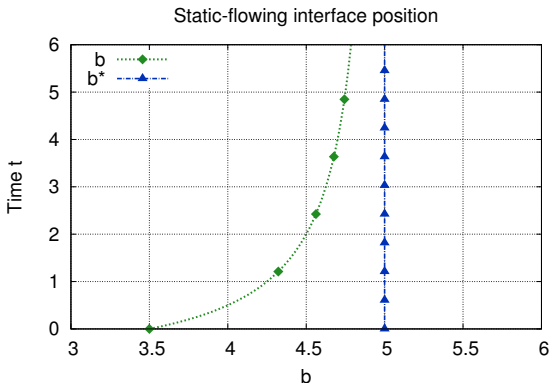**Proposition [Lusso, Bouchut, Ern, Mangeney, 2016]**
With the previous assumptions, assume that

$$b^*(t) \text{ is nondecreasing}, \quad b^0 < b^*(0).$$

Then there is a unique solution with initial data u^0 (that has an interface at b^0) such that $b(t)$ is nondecreasing, continuous and $b(0) = b^0$. Moreover it satisfies

$$b(t) < b^*(t) \text{ for all } t \in [0, T].$$

There is a progressive stopping.

Static-flowing interface position

Proposition [Lusso, Bouchut, Ern, Mangeney, 2016]
With the previous assumptions, assume that
$$b^*(t) \text{ is decreasing}, \quad b^0 < b^*(0).$$
Then there is a unique solution with initial data u^0 (that has an interface at b^0) such that $b(t)$ is continuous in $[0, T]$, piecewise C^1 and $b(0) = b^0$.
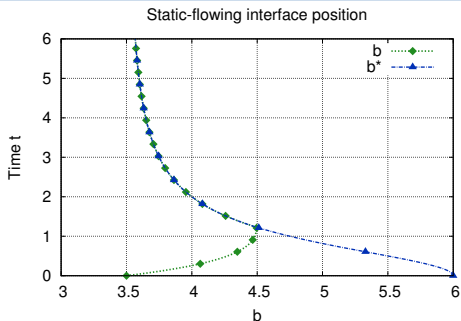
Moreover $b(t)$ is increasing in $[0, t^*]$, and one of the two following cases occurs
(i) $b(t) < b^*(t)$ for all $t \in [0, T]$ and $t^* = T$.
(ii) $b(t)$ reaches $b^*(t)$ at the time t^*, $\dot{b}(t^* - 0) = 0$,
and for all $t \in [t^*, T]$ one has $b(t) = b^*(t)$ (thus b is decreasing in $[t^*, T]$).
There is a progressive stopping, then a progressive starting.

Static-flowing interface position

Dynamics with Z-dependent source term

The proof relies on integrating the differential equation $\partial_t U = -S$ until reaching a boundary of the domain $Z > b(t)$.
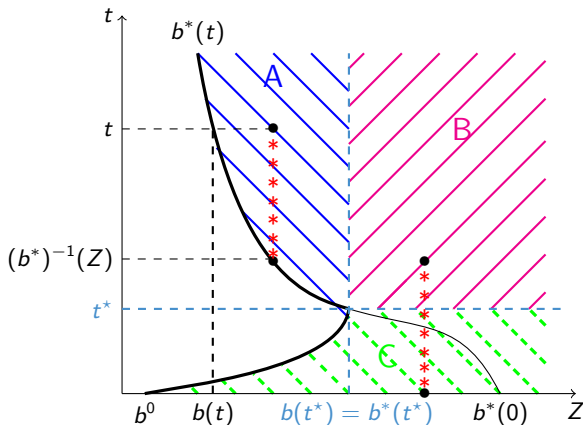

FIG.: Integration of the ordinary differential equation along vertical lines

Proposition [Lusso, Bouchut, Ern, Mangeney, 2016]
With the previous assumptions, assume that

$$b^*(t) \text{ is increasing}, \quad b^0 > b^*(0).$$

Then there is a unique solution with initial data u^0 (that has an interface at b^0) such that $b(t)$ is nondecreasing, continuous and $b(0+) = b^*(0) < b^0$. Moreover

$$b(t) < b^*(t) \text{ for all } t \in (0, T].$$

There is an instantaneous starting of a part of the mass, then a progressive stopping.

Static-flowing interface position
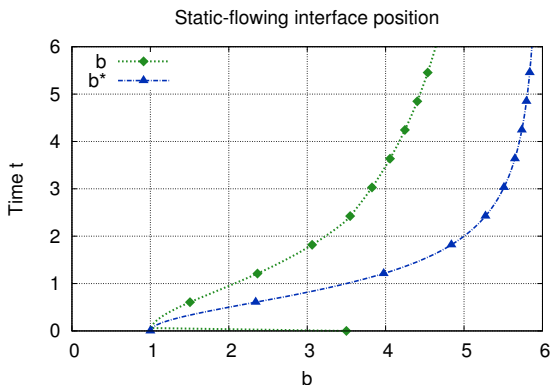
6. Conclusion

- We have established reduced equations for the evolution of the static/flowing interface from 2D viscoplastic models.

- Thin layer and several assumptions are necessary, but the normal variable Z remains, it is not a depth-averaged model.

- The model describes several physically relevant solutions (fully static configurations, steady flow with static/flowing transitions, both characterized by a condition on the slope of the free surface)

- The model generalizes the BCRE model without viscosity. With viscosity it is more complicate and involves a third-order derivative of the velocity.

- A simple numerical method is possible when the X dependency is removed.

- The viscosity enables to describe an initial erosion of the bed that is observed in experiments with initially static bed of the same material.

- The effect of the source term (that represents the feedback via the nonhydrostatic pressure) is to initiate starting or stopping of the mass.

Open issues :

▷ Simulate the coupled problem with X (and Z) dependency, and understand the effect of X inhomogeneities.

▷ Establish a "depth-averaged model" that eliminates the variable Z but retains only the leading effects (evolution of the interface and the slope for example).