A compressible two-layer model for transient gas-liquid flows in pipes

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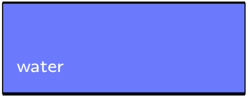
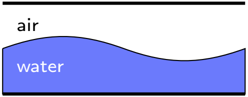
Industrial context

- In nuclear power plants, hydraulic systems are used :

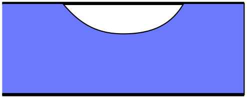


Pressurized flow

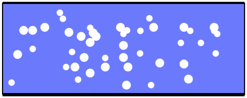
- BUT unwanted air pockets could give rise to transient multi-regime air/water flows :



Stratified flow



Entrapped air pockets



Bubbly flow

- Requirement : determining the maximum acceptable volume of air in pipes to guarantee operability of pumps
  - Experimental studies : how the pump is affected by the different regimes ?
  - **Modeling/Numerical studies : how air pockets will be transported until the pump ?**
- Same issues in wastewater pipes, petroleum industries...

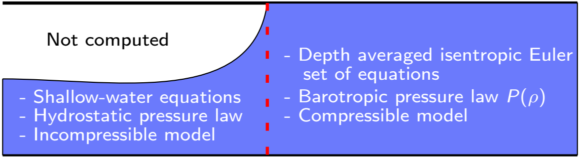
Technical specifications

Our goal is to develop a 1D model satisfying the following constraints :

- **Physical constraints**
  - Two-phase flow, gas and liquid
  - Interactions between both phases
  - Several regimes : pressurized, stratified and entrapped air pockets
  - Transition from stratified to pressurized flow
- **Geometric constraints**
  - Sloping pipes
  - Rectangular channels or circular pipes
- **Mathematical constraints**
  - Hyperbolicity
  - Positivity
  - Entropy inequality
  - Uniqueness of jump conditions

Some available models

- **PFS model** by Bourdarias & Gerbi 07'



Two-equation system
+ Hyperbolic
+ Transition
− Air phase not computed

- **Two-layer model** by Bourdarias, Ersoy & Gerbi 13'

Four-equation system
+ Air phase computed
− Not hyperbolic
− No transition

- **Two-pressure model** by Ransom & Hicks 83'

Five-equation system
+ Hyperbolic
+ Air phase computed
− No transition
− No gravity effects

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- Shallow-water equations
- Hydrostatic pressure law
- Incompressible model

- Depth averaged isentropic Euler set of equations
- Barotropic pressure law $P(\rho)$
- Compressible model

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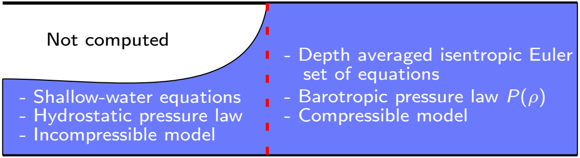

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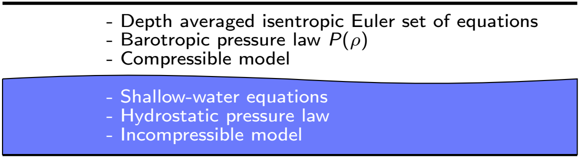

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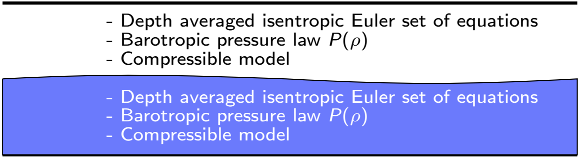

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Proposal

- Development of a 1D compressible two-layer model following the Ransom & Hicks' approach :
  - Isentropic Euler set of equations with a barotropic pressure law $P(\rho)$ for both phases
  - Depth averaging process
- Adding physical constraints :
  - Hydrostatic constraint for water pressure gradient (gravity effects)
  - Transition to the pressurized regime
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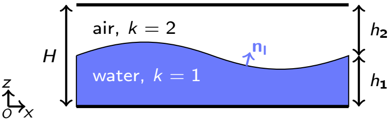
Local governing equations



Isentropic Euler set of equations along x :

$$\frac{\partial \rho_k}{\partial t} + \frac{\partial \rho_k u_k}{\partial x} + \frac{\partial \rho_k w_k}{\partial z} = M_k,$$
$$\frac{\partial \rho_k u_k}{\partial t} + \frac{\partial \rho_k u_k^2}{\partial x} + \frac{\partial \rho_k u_k w_k}{\partial z} + \frac{\partial P_k(\rho_k)}{\partial x} = D_k,$$

with the hydrostatic constraint : $\frac{\partial P_1(\rho_1)}{\partial z} = -\rho_1 g$.

- Pressure laws : Isentropic stiffened gas law for water, perfect gas law for air.
- Interfacial kinetic boundary condition :

$$\frac{\partial h_1}{\partial t} + U_I \frac{\partial h_1}{\partial x} = W_I, \text{ with } \mathbf{u}_I = \begin{pmatrix} U_I \\ W_I \end{pmatrix} \text{ the interfacial velocity.}$$

- Continuity of pressure at the interface : $P_1(x, z = h_1^-, t) = P_2(x, z = h_1^+, t) = P_I$.
- Continuity of the normal velocity at the interface :

$$(\mathbf{u}_1(x, z = h_1^-, t) - \mathbf{u}_2(x, z = h_1^+), t).\mathbf{n}_I = 0.$$

- Impermeability on the walls : $w_1(z=0) = w_2(z=H) = 0$.
- Closure law for the interfacial velocity :

$$\mathbf{u}_I(x,t) = \beta \mathbf{u}_1(x, z = h_1^-, t) + (1-\beta)\mathbf{u}_2(x, z = h_1^+, t), \ \beta \in [0,1].$$

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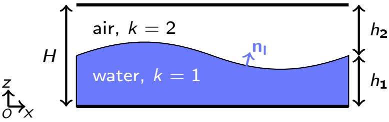

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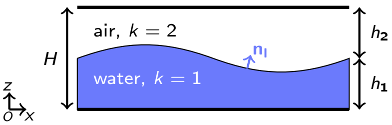

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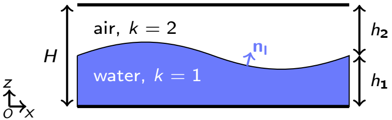

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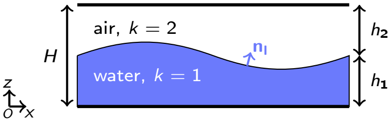

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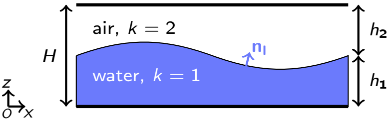

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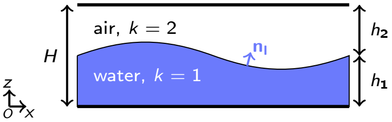

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Averaging process

- Averaging operators :

$$\overline{f_k(x,t)} = \frac{1}{h_k}\int_{z_k}^{h_k+z_k} f_k(x,z,t)dz \text{ with } \begin{cases} z_1 = 0, \\ z_2 = h_1. \end{cases}$$

and

$$\widehat{f_k(x,t)} = \frac{\overline{\rho_k f_k}}{\overline{\rho_k}}$$

- Averaged mass conservation equations :

$$\frac{\partial h_k \overline{\rho_k}}{\partial t} + \frac{\partial h_k \overline{\rho_k}\ \widehat{u_k}}{\partial x} = h_k \overline{M_k}.$$

- Averaged momentum conservation equations with the closure law $\overline{\rho_k u_k^2} = \overline{\rho_k}\ \widehat{u_k}^2$:

$$\frac{\partial h_k \overline{\rho_k}\ \widehat{u_k}}{\partial t} + \frac{\partial h_k (\overline{\rho_k}\ \widehat{u_k}^2 + \overline{P_k}(\overline{\rho_k}))}{\partial x} - P_I \frac{\partial h_k}{\partial x} = h_k \overline{D_k}.$$

- **How to satisfy the hydrostatic constraint for water ?**

$$\frac{\partial P_1(\rho_1)}{\partial z} = -\rho_1 g.$$

Hydrostatic constraint

- Starting point :

$$\frac{\partial P_1(\rho_1)}{\partial z} = -\rho_1 g \Rightarrow P_1(\rho_1) = P_I(x,t) + \int_z^{h_1} \rho_1(x,s,t) g ds.$$

- An integration between 0 and h_1 yields :

$$\int_0^{h_1} P_1(\rho_1) dz = h_1 P_I(x,t) + g \int_0^{h_1} \Big(\int_z^{h_1} \rho_1(x,s,t) ds \Big) dz.$$

- An integration by part of the double integral provides :

$$\int_0^{h_1} P_1(\rho_1) dz = h_1 P_I + g \left[z \int_z^{h_1} \rho_1 ds \right]_0^{h_1} + g \int_0^{h_1} \rho_1 z dz,$$
$$= h_1 P_I + \overline{\rho_1 z} g h_1.$$

- With the closure law $\overline{\rho_1 z} = \overline{\rho_1}\ \overline{z} = \overline{\rho_1} \frac{h_1}{2}$, it comes :

$$\overline{P_1} = P_I + \overline{\rho_1} g \frac{h_1}{2}.$$

  The obtained averaged hydrostatic constraint closes P_I.

- The momentum conservation equation for water under hydrostatic constraint writes :

$$\frac{\partial h_1 \overline{\rho_1}\ \widehat{u_1}}{\partial t} + \frac{\partial}{\partial x} h_1 (\overline{\rho_1}\ \widehat{u_1}^2 + \overline{P_1}) - (\overline{P_1} - \overline{\rho_1} g \frac{h_1}{2}) \frac{\partial h_1}{\partial x} = h_1 \overline{D_1}.$$

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- An integration by part of the double integral provides :

$$\begin{aligned} \int_0^{h_1} P_1(\rho_1) dz &= h_1 P_I + g \left[z \int_z^{h_1} \rho_1 ds \right]_0^{h_1} + g \int_0^{h_1} \rho_1 z dz, \\ &= h_1 P_I + \overline{\rho_1 z} g h_1. \end{aligned}$$

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$$\frac{\partial h_1 \overline{\rho_1}\ \widehat{u_1}}{\partial t} + \frac{\partial}{\partial x} h_1 (\overline{\rho_1}\ \widehat{u_1}^2 + \overline{P_1}) - (\overline{P_1} - \overline{\rho_1} g \frac{h_1}{2}) \frac{\partial h_1}{\partial x} = h_1 \overline{D_1}.$$

Hydrostatic constraint

- Starting point :

$$\frac{\partial P_1(\rho_1)}{\partial z} = -\rho_1 g \Rightarrow P_1(\rho_1) = P_I(x,t) + \int_z^{h_1} \rho_1(x,s,t) g ds.$$

- An integration between 0 and h_1 yields :

$$\int_0^{h_1} P_1(\rho_1) dz = h_1 P_I(x,t) + g \int_0^{h_1} \left(\int_z^{h_1} \rho_1(x,s,t) ds \right) dz.$$

- An integration by part of the double integral provides :

$$\begin{aligned} \int_0^{h_1} P_1(\rho_1) dz &= h_1 P_I + g \left[z \int_z^{h_1} \rho_1 ds \right]_0^{h_1} + g \int_0^{h_1} \rho_1 z dz, \\ &= h_1 P_I + \overline{\rho_1 z} g h_1. \end{aligned}$$

- With the closure law $\overline{\rho_1 z} = \overline{\rho_1}\ \overline{z} = \overline{\rho_1} \frac{h_1}{2}$, it comes :

$$\overline{P_1} = P_I + \overline{\rho_1} g \frac{h_1}{2}.$$

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Resulting averaged system

- **Omitting the operator notations**, the five-equation system corresponding to the five unknowns $(h_1, \rho_1, \rho_2, u_1, u_2)$ writes :

$$
\begin{aligned}
&\frac{\partial h_1}{\partial t} + U_I \frac{\partial h_1}{\partial x} = \Phi_1, \\
&\frac{\partial h_k \rho_k}{\partial t} + \frac{\partial h_k \rho_k u_k}{\partial x} = M_k, \\
&\frac{\partial h_1 \rho_1 u_1}{\partial t} + \frac{\partial h_1(\rho_1 u_1^2 + P_1(\rho_1))}{\partial x} - \left(P_1(\rho_1) - \rho_1 g \frac{h_1}{2}\right) \frac{\partial h_1}{\partial x} = D_1, \\
&\frac{\partial h_2 \rho_2 u_2}{\partial t} + \frac{\partial h_2(\rho_2 u_2^2 + P_2(\rho_2))}{\partial x} - \left(P_1(\rho_1) - \rho_1 g \frac{h_1}{2}\right) \frac{\partial h_2}{\partial x} = D_2,
\end{aligned}
$$

with $h_2 = H - h_1$.

- Closure laws for U_I and the source terms ?
  - Assumption : $U_I = \beta u_1 + (1-\beta) u_2, \ \beta \in [0,1]$.
  - Entropy inequality.

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Entropy inequality and closure laws

- Define

$$E_{c,k} = \frac{1}{2} h_k \rho_k u_k^2$$
kinetic energy of phase k

$$E_{t,k} = h_k \rho_k \Psi_k(\rho_k)$$
thermodynamic energy of phase k
with $\Psi_k'(\rho_k) = \frac{P_k(\rho_k)}{\rho_k^2}$

$$E_{p,1} = \frac{1}{2} \rho_1 g h_1^2$$
gravitational energy of phase 1

and

$$E_1 = E_{c,1} + E_{t,1} + E_{p,1}$$
total energy of phase 1

$$E_2 = E_{c,2} + E_{t,2}$$
total energy of phase 2

- Adding all the contributions, the total energy equation writes

$$\frac{\partial}{\partial t}(E_1 + E_2) + \frac{\partial}{\partial x}(u_1 E_1 + u_2 E_2 + u_1 h_1 P_1 + u_2 h_2 P_2) + f_1 \frac{\partial h_1}{\partial x} = f_2 D_1 + f_3 \Phi_1 + f_4 M_1,$$

with f_i some given functions depending on the state variable.

- Set $f_1 = 0$ to get a conservative form :

$$f_1 = \beta(P_2 - P_1 - \rho_1 g \frac{h_1}{2}) = 0 \Rightarrow \beta = 0 \text{ and } U_I = u_2$$

- Set $D_1 = -f_2$, $\Phi_1 = -f_3$, $M_1 = -f_4$ to get an entropy inequality.

Entropy inequality and closure laws

- Define

$E_{c,k} = \frac{1}{2} h_k \rho_k u_k^2$ kinetic energy of phase k

$E_{t,k} = h_k \rho_k \Psi_k(\rho_k)$ thermodynamic energy of phase k with $\Psi_k'(\rho_k) = \frac{P_k(\rho_k)}{\rho_k^2}$

$E_{p,1} = \frac{1}{2} \rho_1 g h_1^2$ gravitational energy of phase 1

and

$E_1 = E_{c,1} + E_{t,1} + E_{p,1}$ total energy of phase 1

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$$E_{c,k} = \frac{1}{2} h_k \rho_k u_k^2 \qquad E_{t,k} = h_k \rho_k \Psi_k(\rho_k) \qquad E_{p,1} = \frac{1}{2}\rho_1 g h_1^2$$

kinetic energy of phase k — thermodynamic energy of phase k — gravitational energy of phase 1

with $\Psi'_k(\rho_k) = \frac{P_k(\rho_k)}{\rho_k^2}$

and

$$E_1 = E_{c,1} + E_{t,1} + E_{p,1} \qquad E_2 = E_{c,2} + E_{t,2}$$

total energy of phase 1 — total energy of phase 2

- Adding all the contributions, the total energy equation writes

$$\frac{\partial}{\partial t}(E_1 + E_2) + \frac{\partial}{\partial x}(u_1 E_1 + u_2 E_2 + u_1 h_1 P_1 + u_2 h_2 P_2) + f_1 \frac{\partial h_1}{\partial x} = f_2 D_1 + f_3 \Phi_1 + f_4 M_1,$$

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with f_i some given functions depending on the state variable.

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$$f_1 = \beta(P_2 - P_1 - \rho_1 g \frac{h_1}{2}) = 0 \Rightarrow \beta = 0 \text{ and } U_I = u_2$$

- Set $D_1 = -f_2$, $\Phi_1 = -f_3$, $M_1 = -f_4$ to get an entropy inequality.

Entropy inequality and closure laws

- One obtains :

$$\Phi_1 = \lambda_p(P_1 - \rho_1 g \frac{h_1}{2} - P_2) = \lambda_p(P_I - P_2),$$

$$M_k = (-1)^k \lambda_m \Big((\frac{P_1 + \rho_1 g \frac{h_1}{2}}{\rho_1} + \Psi_1) - (\frac{P_2}{\rho_2} + \Psi_2) + \frac{u_2^2 - u_1^2}{2} \Big),$$

$$D_k = (-1)^k \lambda_u (u_1 - u_2).$$

where λ_p, λ_m and λ_u are positive bounded functions which depend on the state variable $(h_1, \rho_1, \rho_2, u_1, u_2)$.

- The entropy inequality thus writes :

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial \mathcal{G}}{\partial x} \leq 0$$

where the entropy $\mathcal{E}$ and the entropy flux $\mathcal{G}$ are defined by

$$\mathcal{E} = E_{c,1} + E_{p,1} + E_{t,1} + E_{c,2} + E_{t,2},$$

$$\mathcal{G} = u_1(E_{c,1} + E_{p,1} + E_{t,1}) + u_2(E_{c,2} + E_{t,2}) + u_1 h_1 P_1 + u_2 h_2 P_2,$$

with

$$E_{c,k} = \frac{1}{2} h_k \rho_k u_k^2, \; E_{t,k} = h_k \rho_k \Psi_k(\rho_k), \; E_{p,1} = \rho_1 g \frac{h_1^2}{2}.$$

Resulting averaged closed system

1D compressible two-layer model for air/water flows

$$
\begin{aligned}
&\frac{\partial h_1}{\partial t} + u_2 \frac{\partial h_1}{\partial x} = \lambda_p (P_1 - \rho_1 g \frac{h_1}{2} - P_2) = \lambda_p (P_I - P_2), \\
&\frac{\partial h_1 \rho_1}{\partial t} + \frac{\partial h_1 \rho_1 u_1}{\partial x} = 0, \\
&\frac{\partial h_2 \rho_2}{\partial t} + \frac{\partial h_2 \rho_2 u_2}{\partial x} = 0, \\
&\frac{\partial h_1 \rho_1 u_1}{\partial t} + \frac{\partial h_1 (\rho_1 u_1^2 + P_1)}{\partial x} - (P_1 - \rho_1 g \frac{h_1}{2}) \frac{\partial h_1}{\partial x} = \lambda_u (u_2 - u_1), \\
&\frac{\partial h_2 \rho_2 u_2}{\partial t} + \frac{\partial h_2 (\rho_2 u_2^2 + P_2)}{\partial x} - (P_1 - \rho_1 g \frac{h_1}{2}) \frac{\partial h_2}{\partial x} = \lambda_u (u_1 - u_2).
\end{aligned}
$$

Consistency with the shallow water equations

- Pressure relaxation. Consider the following system :

$$\frac{\partial h_1}{\partial t} = \lambda_p(P_I - P_2)$$
$$\frac{\partial h_k \rho_k}{\partial t} = 0.$$

Easy calculations yield :

$$P_I \underset{t \to +\infty}{\longrightarrow} P_2.$$

- Without source terms, using $P_I = P_1 - \rho_1 g \frac{h_1}{2}$, mass and momentum conservation equations for water writes :

$$\frac{\partial h_1 \rho_1}{\partial t} + \frac{\partial h_1 \rho_1 u_1}{\partial x} = 0,$$
$$\frac{\partial h_1 \rho_1 u_1}{\partial t} + \frac{\partial h_1 \rho_1 u_1^2}{\partial x} + \frac{\partial \rho_1 g \frac{h_1^2}{2}}{\partial x} + h_1 \frac{\partial P_I}{\partial x} = 0.$$

- Asymptotically, it may be read as a compressible shallow water system with variable atmospheric pressure $P_2(x, t)$.

Comments on the closed system

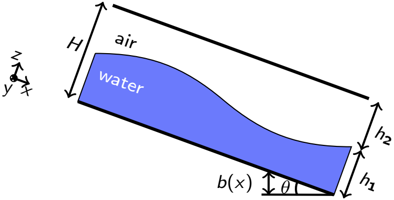
- **Comparison with the two-fluid two-pressure models.**
  - Same structure as a Baer-Nunziato type model obtained with statistical averaging.
  - Statistical framework : $\beta \in \{0, 1, \frac{h_1\rho_1}{h_1\rho_1 + h_2\rho_2}\}$.
  - Significant mathematical properties.

- **Towards pressurized flows.**
  Formally, considering $h_1 = H$ one obtains :

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_1 u_1}{\partial x} = 0,$$
$$\frac{\partial \rho_1 u_1}{\partial t} + \frac{\partial (\rho_1 u_1^2 + P_1(\rho_1))}{\partial x} = 0,$$

  as soon as the source terms vanish when $h_1 = H$.

- **Sloping pipes**



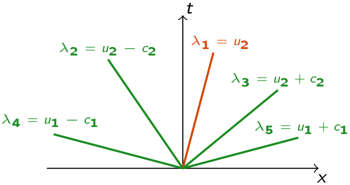
Constant slope : replace g by $g\cos\theta$ in the closure laws.

Variable slope : curvilinear formulation ?

Mathematical properties : hyperbolicity

We study the homogeneous system :

$$\frac{\partial h_1}{\partial t} + u_2 \frac{\partial h_1}{\partial x} = 0,$$
$$\frac{\partial h_k \rho_k}{\partial t} + \frac{\partial h_k \rho_k u_k}{\partial x} = 0,$$
$$\frac{\partial h_k \rho_k u_k}{\partial t} + \frac{\partial h_k(\rho_k u_k^2 + P_k)}{\partial x} - (P_1 - \rho_1 g \frac{h_1}{2}) \frac{\partial h_k}{\partial x} = 0.$$



Hyperbolicity

The homogeneous system is hyperbolic under the non-resonant condition :

$$|u_1 - u_2| \neq c_1.$$

Its eigenvalues are unconditionally real and given by

$$\lambda_1 = u_2,$$
$$\lambda_2 = u_2 - c_2, \ \lambda_3 = u_2 + c_2,$$
$$\lambda_4 = u_1 - c_1, \ \lambda_5 = u_1 + c_1.$$

- Remarks :
  - The eigenstructure can be easily detailed
  - In the context of air/water flows, $c_1 \approx 1500 \ m.s^{-1}$ and the resonant situation is clearly out of the scope of interest.

Mathematical properties : characteristic fields

Nature of characteristic fields and Riemann invariants

The field associated with the 1-wave λ_1 is linearly degenerate.

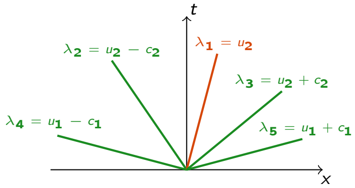
The fields associated with the p-waves $\lambda_p,\ p = 2, .., 5,$ are genuinely nonlinear.

If I^P denotes the vector of p-Riemann invariants, one obtains :

$$I^1 = \Big(u_2, m_1(u_1-u_2), m_1 u_1(u_1-u_2) + h_1 P_1 + h_2 P_2, \frac{P_1 + \rho_1 g \frac{h_1}{2}}{\rho_1} + \Psi_1 + \frac{(u_1-u_2)^2}{2}\Big),$$

$$I^2 = \Big(h_1, \rho_1, u_1, u_2 + \int \frac{c_2(\rho_2)}{\rho_2} d\rho_2\Big),\ I^3 = \Big(h_1, \rho_1, u_1, u_2 - \int \frac{c_2(\rho_2)}{\rho_2} d\rho_2\Big),$$

$$I^4 = \Big(h_1, \rho_2, u_2, u_1 + \int \frac{c_1(\rho_1)}{\rho_1} d\rho_1\Big),\ I^5 = \Big(h_1, \rho_2, u_2, u_1 - \int \frac{c_1(\rho_1)}{\rho_1} d\rho_1\Big),$$



Mathematical properties : jump conditions

Jump conditions for genuine non-linear fields

For all genuine non-linear fields corresponding to the p-waves, $p = 2, ..., 5$, the Rankine-Hugoniot jump conditions across a single discontinuity of speed σ write

$$[h_k] = 0,$$
$$[m_k(u_k - \sigma)] = 0,$$
$$[m_k u_k(u_k - \sigma) + h_k P_k] = 0,$$

where brackets [.] denote the difference between the states on both sides of the discontinuity.

- Across the linearly degenerate field :
  - h_1 may jump.
  - The non-conservative products $u_2 \partial_x h_1$ and $(P_1 - \rho_1 g \frac{h_1}{2})\partial_x h_1$ are well defined using the available 1-Riemann invariants.
- One can build analytical solutions including rarefaction waves, shock waves and contact discontinuities.

Mathematical properties : positivity

Positivity

Let L and T be two positive and real constants. Assume that u_k, $\partial_x u_k$ and the source terms belong to $L^\infty([0,L]\times[0,T])$ for $k=1,2$. Then, admissible inlet boundary conditions lead to

$$h_k(t,x)\in[0,H],\quad \forall(x,t)\in[0,L]\times[0,T],$$
$$\rho_k(t,x)\geq 0,\quad \forall(x,t)\in[0,L]\times[0,T],$$

when restricting ourselves to regular solutions.

- *Classical* result considering regular solutions of the equation :

$$\frac{\partial h_1}{\partial t}+u_2\frac{\partial h_1}{\partial x}=\Phi_1,$$
$$\frac{\partial h_k\rho_k}{\partial t}+u_k\frac{\partial h_k\rho_k}{\partial x}+h_k\rho_k\frac{\partial u_k}{\partial x}=M_k.$$

Conclusion

- This 1D compressible two-layer model satisfies :
  - Physical constraints
    - Gas-liquid model
    - Several regimes : stratified, pressurized, entrapped air pockets (formally)
    - Transition between the regimes (formally)
  - Geometric constraints
    - Sloping pipes (constant angle only)
    - Rectangular channels and circular pipes with variable cross-section
  - Mathematical constraints
    - Hyperbolicity
    - Positivity of h_k and ρ_k
    - Entropy inequality
    - Uniqueness of jump conditions

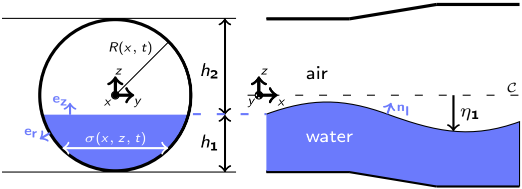
- The current work involves :
  - Development of a numerical solver
    - With large time steps : implicit-explicit schemes to avoid the acoustic CFL condition
    - Robust enough to deal with vanishing phases, $h_k \to 0$
  - Validation against experimental data

Thank you for your attention

Contact : charles.demay@edf.fr

- Submitted to *Continuum Mechanics and Thermodynamics* journal :
  A compressible two-layer model for transient gas-liquid flows in pipes.
  C. Demay, J.-M. Hérard (dec. 2015).

Circular pipe with variable cross-section



1D compressible two-layer model for air/water flows in pipe with variable cross-section

$$\frac{\partial A_1}{\partial t} + u_2 \frac{\partial A_1}{\partial x} = \lambda_p (P_1 - \rho_1 g \ell_1 - P_2) + \frac{\theta_k}{2\pi} \left(\frac{\partial S}{\partial t} + u_2 \frac{\partial S}{\partial x} \right),$$

$$\frac{\partial A_k \rho_k}{\partial t} + \frac{\partial A_k \rho_k u_k}{\partial x} = 0,$$

$$\frac{\partial A_1 \rho_1 u_1}{\partial t} + \frac{\partial A_1 (\rho_1 u_1^2 + P_1)}{\partial x} - (P_1 - \rho_1 g \ell_1) \frac{\partial A_1}{\partial x} = \lambda_u (u_2 - u_1) + \rho_1 g \ell_1 \frac{\theta_1}{2\pi} \frac{\partial S}{\partial x},$$

$$\frac{\partial A_2 \rho_2 u_2}{\partial t} + \frac{\partial A_2 (\rho_2 u_2^2 + P_2)}{\partial x} - (P_1 - \rho_1 g \ell_1) \frac{\partial A_2}{\partial x} = \lambda_u (u_1 - u_2) + (P_2 - (P_1 - \rho_1 g \ell_1)) \frac{\theta_2}{2\pi} \frac{\partial S}{\partial x}.$$