A two-phase solid/fluid model for dense granular flows including dilatancy effects. Part II

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Outline

Jackson's model

The two mass conservation equations for the solid and fluid phases are, respectively,

$$\begin{aligned} \partial_t(\rho_s\varphi) + \nabla\cdot(\rho_s\varphi v) &= 0, \\ \partial_t(\rho_f(1-\varphi)) + \nabla\cdot(\rho_f(1-\varphi)u) &= 0, \end{aligned}$$

and equations of momentum conservation for each phase are

$$\begin{aligned} \rho_s\varphi(\partial_t v + (v\cdot\nabla)v) &= -\nabla\cdot T_s + f_0 + \rho_s\varphi\,\mathbf{g}, \\ \rho_f(1-\varphi)(\partial_t u + (u\cdot\nabla)u) &= -\nabla\cdot T_{f_m} - f_0 + \rho_f(1-\varphi)\mathbf{g}. \end{aligned}$$

- The velocities are: v for the solid phase, u for the fluid phase,
- The (symmetric) stress tensors are: T_s for the solid, T_{f_m} for the fluid.
- The constant densities are denoted by: ρ_s for the solid, ρ_f for the fluid.
- The force f_0 is decomposed into the sum of the buoyancy force and all remaining contributions f:

$$f_0 = -\varphi\nabla p_{f_m} + f.$$

- The solid volume fraction is φ.
- Closure:

$$\nabla\cdot v = \Phi.$$

Domain

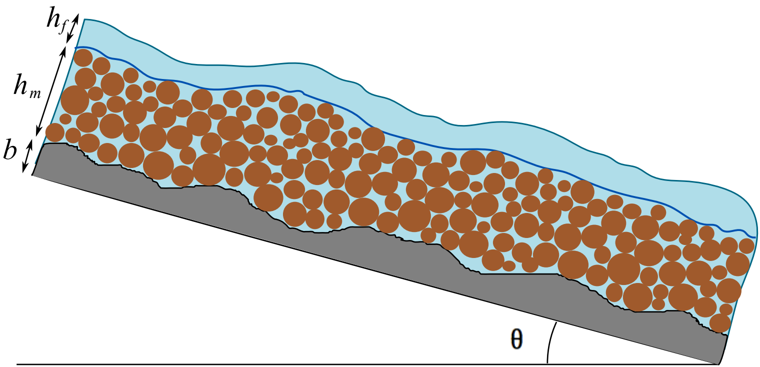


Figure: Domain and geometrical parameters.

- The solid-fluid mixture lies between a fixed bottom and an upper pure fluid layer.
- h_m is the height of the mixture layer.
- h_f is the height of the pure fluid layer over the mixture.

Boundary conditions:

- **At the bottom:**
  - Non penetration condition:

$$u \cdot n = 0, \qquad v \cdot n = 0$$

  where n is the upward space unit normal (i.e. the normal to the topography).

  - Coulomb friction law:

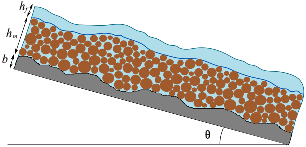
$$(T_s n)_\tau = -\tan \delta_{\text{eff}} \operatorname{sgn}(v)(T_s n) \cdot n,$$

  where δ_{eff} is the effective intergranular Coulomb friction angle.

  - Navier friction condition for the fluid phase:

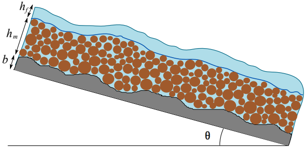
$$(T_{f_m} n)_\tau = -k_b u,$$

  for some coefficient $k_b \geq 0$.



Boundary conditions:

- **At the free surface:**
  - No tension for the fluid
  $$T_f N_X = 0.$$
  - A kinematic condition:
  $$N_t + u_f \cdot N_X = 0.$$
  where $N = (N_t, N_X)$ is a time-space normal to the free surface.



Boundary conditions:

- **At the interface:**
  - A kinematic condition for the solid phase,

    $$\tilde{N}_t + v \cdot \tilde{N}_X = 0.$$

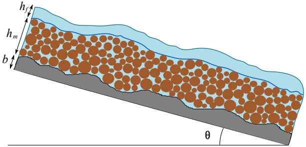
    where we denote by $\tilde{N} = (\tilde{N}_t, \tilde{N}_X)$ a time-space upward normal to the interface.

  - A Navier fluid friction condition

    $$\left(\frac{T_{fm} + T_f}{2} \tilde{N}_X \right)_\tau = -k_i (u_f - u)_\tau .$$

    where $k_i \geq 0$ is a friction coefficient.

  - **Additional jump relations have to be prescribed**. These relations state that the fluxes on both sides of the interface are related through transfer conditions. These are determined by global conservation properties, under the form of Rankine-Hugoniot conditions.



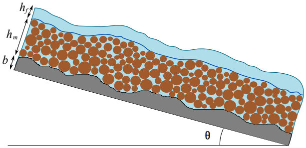
Jump conditions at the interface

- We must first ensure that the **total fluid mass is conserved**:

$$\tilde{N}_t + u_f \cdot \tilde{N}_X = (1 - \varphi^*)(\tilde{N}_t + u \cdot \tilde{N}_X) \equiv \mathcal{V}_f,$$

where:

- φ^* is the value of the solid volume fraction at the interface.
- The term $\mathcal{V}_f$ defines the fluid mass that is transferred from the mixture to the fluid-only layer ($\mathcal{V}_f < 0$ means that the fluid is transferred from the fluid-only region to the mixture region).



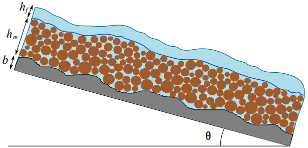
Jump conditions at the interface

- **Conservation of the total momentum** gives

$$\rho_f \mathcal{V}_f (u - u_f) + (T_s + T_{f_m})\tilde{N}_X = T_f \tilde{N}_X.$$

- **The energy balance** through the interface yields the stress transfer condition:

$$T_s \tilde{N}_X = \left(\frac{\rho_f}{2} \left((u - u_f) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|} \right)^2 + \left((T_{f_m} \tilde{N}_X) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|^2} - p_{f_m} \right) \frac{\varphi^*}{1 - \varphi^*} \right) \tilde{N}_X.$$



Asymptotic hypothesis

If $\epsilon = H/L$, where: H and L are the characteristic width and length of the domain, respectively,

$$h_m \sim \epsilon,\ h_f \sim \epsilon,\ \nabla_{\mathbf{x}} b = O(\epsilon),\ T_s = O(\epsilon),\ T_{f_m} = O(\epsilon),\ T_f = O(\epsilon),$$
$$v^{\mathbf{x}} = O(1),\ u^{\mathbf{x}} = O(1),\ u_f^{\mathbf{x}} = O(1),\ \varphi = O(1),\ \Phi = O(1),$$
$$k_b = O(\epsilon),\ k_i = O(\epsilon).$$

- Taking L as typical length unit, $\tau = \sqrt{L/g}$ as typical time unit,
- Then, all the natural units can be expressed in terms of L, τ, and ρ_s (or ρ_f).

- We assume that:
  - The unknowns vary at the scales L in the downslope direction,
  - ϵL in the normal direction,
  - and τ in time,

  which means formally that

$$\nabla_{\mathbf{x}} = O(1), \quad \partial_z = O(\epsilon^{-1}), \quad \partial_t = O(1).$$

Asymptotic hypothesis

$$\partial_t h_f + \nabla_{\mathbf{x}} \cdot \int_{b+h_m}^{b+h_m+h_f} u_f^{\mathbf{x}} dz = \mathcal{V}_f.$$

Then,

$$\mathcal{V}_f = O(\epsilon).$$

$$h_m \sim \epsilon,\ h_f \sim \epsilon,\ \nabla_{\mathbf{x}} b = O(\epsilon),\ T_s = O(\epsilon),\ T_{fm} = O(\epsilon),\ T_f = O(\epsilon),$$
$$v^{\mathbf{x}} = O(1),\ u^{\mathbf{x}} = O(1),\ u_f^{\mathbf{x}} = O(1),\ \varphi = O(1),\ \Phi = O(1),\ k_b = O(\epsilon),\ k_i = O(\epsilon).$$

Asymptotic hypothesis

As in (Bouchut et al. 2003; Bouchut and Westdickenberg 2004) we shall assume that the tangential velocities and the solid volume fraction do not depend on z up to errors in $O(\epsilon^2)$,

$$v^{\mathbf{x}} = \overline{v^{\mathbf{x}}}(t,\mathbf{x}) + O(\epsilon^2), \tag{3}$$

$$u^{\mathbf{x}} = \overline{u^{\mathbf{x}}}(t,\mathbf{x}) + O(\epsilon^2), \tag{4}$$

$$u_f^{\mathbf{x}} = \overline{u_f^{\mathbf{x}}}(t,\mathbf{x}) + O(\epsilon^2), \tag{5}$$

$$\varphi = \bar{\varphi}(t,\mathbf{x}) + O(\epsilon^2). \tag{6}$$

Then, from

$$\nabla_{\mathbf{x}} \cdot v^{\mathbf{x}} + \partial_z v^z = \Phi.$$

and the non-penetration condition ($v^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} b = v^z$ at $z = b$), we get that $v^z = O(\epsilon)$.

Similarly, from fluid mass phase conservation,

$$\partial_t(1-\varphi) + \nabla_{\mathbf{x}} \cdot \big((1-\varphi)u^{\mathbf{x}}\big) + \partial_z\big((1-\varphi)u^z\big) = 0,$$

and the non-penetration condition we get $(1-\varphi)u^z = O(\epsilon)$, thus $u^z = O(\epsilon)$.

Asymptotic hypothesis

We assume also for the closure function an expansion as

$$\Phi = \bar{\Phi}(t, \mathbf{x}) + O(\epsilon^2),$$

with

$$\bar{\Phi} = K\bar{\dot{\gamma}}(\bar{\varphi} - \bar{\varphi}_c^{eq}).$$

Remark: We adopt this approximation in order to make the derivation possible, even if it looks not appropriate because of the dependency on the pressure of φ_c^{eq}, and of the nonlinear coupling of $\dot{\gamma}$.

Asymptotic hypothesis

The stress tensors T_k ($k = s, f_m, f$), they are decomposed as

$$T_k = p_k \,\mathrm{Id} + \widetilde{T}_k, \tag{7}$$

and suitable rheological assumptions should be made to define $\widetilde{T}_k$.

- Since we aim to represent only depth-average effects, we prefer to simplify the rheologies and replace the effect of the stress tensors inside the domain by boundary layers due to the friction conditions.

- Thus we shall assume that the stresses $\widetilde{T}_k$ are $O(\epsilon^2)$ far from the boundaries $z = b, b + h_m$ and can just be nonzero close to these boundaries.

- We assume that

  $\widetilde{T}_s^{\mathbf{xz}}$, $\widetilde{T}_{fm}^{\mathbf{xz}}$, $\widetilde{T}_f^{\mathbf{xz}}$ can be $O(\epsilon)$ close to the boundaries $z = b, b + h_m$, but are $O(\epsilon^2)$ far from these boundaries,

  while the other components satisfy

$$\widetilde{T}_k^{\mathbf{xx}} = \widetilde{T}_k^{zz} = O(\epsilon^2) \quad \text{everywhere.}$$

Asymptotic hypothesis

The drag term is defined by

$$f = \tilde{\beta}(u - v),$$

$\tilde{\beta}$ being the drag coefficient given by

$$\tilde{\beta} = (1-\varphi)^2 \frac{\eta_f}{\kappa},$$

where η_f is the dynamic viscosity of the fluid and κ is the hydraulic permeability of the granular aggregate, that depends on φ.

We have

$$\tilde{\beta} = \bar{\beta}(t, \mathbf{x})\big(1 + O(\epsilon^2)\big),$$

with

$$\bar{\beta} = (1-\bar{\varphi})^2 \frac{\eta_f}{\bar{\kappa}}.$$

We shall consider two possible sets of assumptions.

Asymptotic hypothesis: drag term

(i) The drag term is quite strong, that is

$$\bar{\beta} \sim \epsilon^{-1}.$$

Then since the drag force $\tilde{\beta}(u - v)$ has to balance gravity terms, it necessarily remains bounded. This implies that

$$u^{\mathbf{x}} - v^{\mathbf{x}} = O(\epsilon).$$

(ii) The drag term is moderate, that is

$$\bar{\beta} = O(1).$$

In this case one has just $u^{\mathbf{x}} - v^{\mathbf{x}} = O(1)$.

Note that in both cases one has $\bar{\beta}(u^{\mathbf{x}} - v^{\mathbf{x}}) = O(1)$

Interface pressure

- The fluid pressure in the fluid-only layer

$$p_f = \rho_f g \cos\theta (b + h_m + h_f - z) + O(\epsilon^2) \quad \text{for } b + h_m < z < b + h_m + h_f$$

- At the interface we have

$$p_{f_m}{}_{|b+h_m} = p_f{}_{|b+h_m} - p_s{}_{|b+h_m} + O(\epsilon^2).$$

- Moreover, the jump conditions for the energy balance through the interface yields the stress transfer condition:

$$T_s \tilde{N}_X = \left(\frac{\rho_f}{2} \left((u - u_f) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|} \right)^2 + \left((T_{f_m} \tilde{N}_X) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|^2} - p_{f_m} \right) \frac{\varphi^*}{1 - \varphi^*} \right) \tilde{N}_X .$$

- We look now for a boundary condition of the form

$$T_s \tilde{N}_X = p_s^* \tilde{N}_X, \qquad (p_s^* = p_s{}_{|b+h_m} + O(\epsilon^2))$$

Then,

$$p_s^* = \left(\frac{\rho_f}{2} \left((u - u_f) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|} \right)^2 + \left((T_{f_m} \tilde{N}_X) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|^2} - p_{f_m} \right) \frac{\varphi^*}{1 - \varphi^*} \right).$$

Interface pressure

That is,

$$p_s^* = \frac{\rho_f}{2}\frac{1}{1+|\nabla_{\mathbf{x}}(b+h_m)|^2}\Big(u^z - u_f^z - (u^{\mathbf{x}} - u_f^{\mathbf{x}})\cdot\nabla_{\mathbf{x}}(b+h_m)\Big)^2 + \frac{\varphi^*}{1-\varphi^*}\Big(\frac{(T_{f\,m}^{\mathbf{xx}}\nabla_{\mathbf{x}}(b+h_m))\cdot\nabla_{\mathbf{x}}(b+h_m) - 2T_{f\,m}^{\mathbf{x}z}\cdot\nabla_{\mathbf{x}}(b+h_m) + T_{f\,m}^{zz}}{1+|\nabla_{\mathbf{x}}(b+h_m)|^2} - p_{f_m}\Big).$$

Thus

$$p_s^* = O(\epsilon^2), \quad p_{s\,|b+h_m} = O(\epsilon^2), \quad p_{f_m\,|b+h_m} = p_{f\,|b+h_m} + O(\epsilon^2).$$

Then we obtain the pressure for the fluid in the mixture at the interface,

$$p_{f_m\,|b+h_m} = \rho_f g \cos\theta h_f + O(\epsilon^2).$$

$$p_{s\,|b+h_m} = O(\epsilon^2)$$

Excess pore pressure

In the mixture, the normal fluid momentum equation gives

$$\partial_z p_{f_m} = -\rho_f g \cos\theta - \frac{\bar{\beta}}{1-\bar{\varphi}}(u^z - v^z) + O(\epsilon).$$

Integrating with respect to z, we obtain for $b < z < b + h_m$

$$p_{f_m} = p_{f_m\,|b+h_m} + \rho_f g \cos\theta(b + h_m - z) + \frac{\bar{\beta}}{1-\bar{\varphi}} \int_z^{b+h_m} (u^z - v^z)(z')dz' + O(\epsilon^2),$$

Then,

$$p_{f_m} = \rho_f g \cos\theta(b + h_m + h_f - z) + p^e_{f_m} + O(\epsilon^2) \quad \text{for } b < z < b + h_m,$$

where

$$p^e_{f_m} \equiv \frac{\bar{\beta}}{1-\bar{\varphi}} \int_z^{b+h_m} (u^z - v^z)(z')dz'$$

is the **excess pore pressure**.

Solid pressure

Moreover, the solid normal momentum equation gives

$$\partial_z p_s = -\bar{\varphi}\partial_z p_{f_m} - \bar{\varphi}\rho_s g \cos\theta + \bar{\beta}(u^z - v^z) + O(\epsilon).$$

Integrating with respect to z gives the expression of the solid pressure,

$$p_s = p_{s\,|b+h_m} - \bar{\varphi}(p_{f_m} - p_{f_m\,|b+h_m}) + \bar{\varphi}\rho_s g \cos\theta(b+h_m-z) - \bar{\beta}\int_z^{b+h_m}(u^z - v^z)(z')dz' + O(\epsilon^2).$$

The solid pressure is given by

$$p_s = \bar{\varphi}(\rho_s - \rho_f) g \cos\theta(b + h_m - z) - p^e_{f_m} + O(\epsilon^2) \quad \text{for } b < z < b + h_m.$$

Note that its nonhydrostatic component is the opposite of that of p_{f_m}

Evaluation of the excess pore pressure

$$p^e_{fm} \equiv \frac{\bar{\beta}}{1-\bar{\varphi}} \int_z^{b+h_m} (u^z - v^z)(z')dz'$$

We have thus to evaluate $u^z - v^z$ up to $O(\epsilon^2)$ errors.

- The closure equation gives:

$$\nabla_{\mathbf{x}} \cdot v^{\mathbf{x}} + \partial_z v^z = \Phi.$$

  By using the non-penetration condition we get

$$v^z = \overline{v^{\mathbf{x}}} \cdot \nabla_{\mathbf{x}} b + (z-b)(\bar{\Phi} - \nabla_{\mathbf{x}} \cdot \overline{v^{\mathbf{x}}}) + O(\epsilon^3).$$

- Next, adding the mass equations in the mixture, we find

$$\nabla_{\mathbf{x}} \cdot (\varphi v^{\mathbf{x}} + (1-\varphi)u^{\mathbf{x}}) + \partial_z(\varphi v^z + (1-\varphi)u^z) = 0,$$

  and using the non-penetration conditions we get

$$\varphi v^z + (1-\varphi)u^z = (\bar{\varphi}\overline{v^{\mathbf{x}}} + (1-\bar{\varphi})\overline{u^{\mathbf{x}}}) \cdot \nabla_{\mathbf{x}} b - (z-b)\nabla_{\mathbf{x}} \cdot (\bar{\varphi}\overline{v^{\mathbf{x}}} + (1-\bar{\varphi})\overline{u^{\mathbf{x}}}) + O(\epsilon^3).$$

- By subtracting previous equations yields

$$u^z - v^z = (\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \cdot \nabla_{\mathbf{x}} b - \frac{z-b}{1-\bar{\varphi}} \Big(\bar{\Phi} + \nabla_{\mathbf{x}} \cdot \big((1-\bar{\varphi})(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}})\big) \Big) + O(\epsilon^3).$$

Evaluation of the excess pore pressure

- $$p^e_{fm} \equiv \frac{\bar{\beta}}{1-\bar{\varphi}} \int_z^{b+h_m} (u^z - v^z)(z')dz'$$

- $$u^z - v^z = (\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \cdot \nabla_{\mathbf{x}} b - \frac{z-b}{1-\bar{\varphi}} \left(\bar{\Phi} + \nabla_{\mathbf{x}} \cdot \left((1-\bar{\varphi})(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \right) \right) + O(\epsilon^3).$$

- Then,

$$p^e_{fm} = \frac{\bar{\beta}}{1-\bar{\varphi}} \Bigg((b+h_m-z)(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \cdot \nabla_{\mathbf{x}} b - \frac{1}{2} \frac{h_m^2 - (z-b)^2}{1-\bar{\varphi}} \left(\bar{\Phi} + \nabla_{\mathbf{x}} \cdot \left((1-\bar{\varphi})(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \right) \right) + O(\epsilon^4) \Bigg).$$

Evaluation of the excess pore pressure

We can then consider two possible sets of expansions for the values of $(p^e_{f_m})_{|b}, \overline{p^e_{f_m}}$:

(i) $(\bar\beta = O(\epsilon^{-1}))$ The values of $(p^e_{f_m})_{|b}, \overline{p^e_{f_m}}$ are given simply by

$$(p^e_{f_m})_{|b} = -\frac{\bar\beta}{(1-\bar\varphi)^2}\frac{h_m^2}{2}\bar\Phi + O(\epsilon^2), \quad \overline{p^e_{f_m}} = -\frac{\bar\beta}{(1-\bar\varphi)^2}\frac{h_m^2}{3}\bar\Phi + O(\epsilon^2).$$

(ii) $(\bar\beta = O(1))$ The values of $(p^e_{f_m})_{|b}, \overline{p^e_{f_m}}$ are given by

$$(p^e_{f_m})_{|b} = \frac{\bar\beta}{1-\bar\varphi}\left(h_m(\overline{u^{\mathbf{x}}}-\overline{v^{\mathbf{x}}})\cdot\nabla_{\mathbf{x}} b - \frac{h_m^2}{2(1-\bar\varphi)}\left(\bar\Phi + \nabla_{\mathbf{x}}\cdot((1-\bar\varphi)(\overline{u^{\mathbf{x}}}-\overline{v^{\mathbf{x}}}))\right)\right) + O(\epsilon^3),$$

$$\overline{p^e_{f_m}} = \frac{\bar\beta}{1-\bar\varphi}\left(\frac{h_m}{2}(\overline{u^{\mathbf{x}}}-\overline{v^{\mathbf{x}}})\cdot\nabla_{\mathbf{x}} b - \frac{h_m^2}{3(1-\bar\varphi)}\left(\bar\Phi + \nabla_{\mathbf{x}}\cdot((1-\bar\varphi)(\overline{u^{\mathbf{x}}}-\overline{v^{\mathbf{x}}}))\right)\right) + O(\epsilon^3).$$

The two-phase two-layer model

- From the mass conservation equations we obtain:

$$\partial_t(\bar{\varphi} h_m) + \nabla_{\mathbf{x}} \cdot (\bar{\varphi} h_m \overline{v^{\mathbf{x}}}) = 0,$$
$$\partial_t\big((1-\bar{\varphi})h_m\big) + \nabla_{\mathbf{x}} \cdot \big((1-\bar{\varphi})h_m \overline{u^{\mathbf{x}}}\big) = -\mathcal{V}_f,$$
$$\partial_t h_f + \nabla_{\mathbf{x}} \cdot (h_f \overline{u^{\mathbf{x}}_f}) = \mathcal{V}_f.$$

- Moreover, the evolution equation for $\bar{\varphi}$ is

$$\partial_t \bar{\varphi} + \overline{v^{\mathbf{x}}} \cdot \nabla_{\mathbf{x}} \bar{\varphi} = -\bar{\varphi}\bar{\Phi}.$$

By combining it with previous equations we obtain

$$\mathcal{V}_f = -h_m \bar{\Phi} - \nabla_{\mathbf{x}} \cdot \Big((1-\bar{\varphi})h_m(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}})\Big).$$

The two-phase tow-layer model

$$
\begin{aligned}
\partial_t h_f + \nabla_{\mathbf{x}} \cdot (h_f \overline{u_f^{\mathbf{x}}}) &= \mathcal{V}_f, \\
\rho_f (\partial_t \overline{u_f^{\mathbf{x}}} + \overline{u_f^{\mathbf{x}}} \cdot \nabla_{\mathbf{x}} \overline{u_f^{\mathbf{x}}}) &= -\rho_f g \cos\theta \nabla_{\mathbf{x}} (b + h_m + h_f) \\
&= -\frac{1}{h_f} \big(\frac{1}{2} \rho_f \mathcal{V}_f + k_i\big) (\overline{u_f^{\mathbf{x}}} - \overline{u^{\mathbf{x}}}) - \rho_f g \sin\theta (1,0)^t, \\
\partial_t \bar{\varphi} + \overline{v^{\mathbf{x}}} \cdot \nabla_{\mathbf{x}} \bar{\varphi} &= -\bar{\varphi} \bar{\Phi},
\end{aligned}
$$

The two-phase tow-layer model

$$
\begin{aligned}
\partial_t(\bar\varphi h_m) + \nabla_{\mathbf{x}}\cdot(\bar\varphi h_m \overline{v^{\mathbf{x}}}) &= 0,\\
\rho_s\bar\varphi(\partial_t\overline{v^{\mathbf{x}}} + \overline{v^{\mathbf{x}}}\cdot\nabla_{\mathbf{x}}\overline{v^{\mathbf{x}}}) &= -\bar\varphi g\cos\theta\big(\rho_s\nabla_{\mathbf{x}}(b+h_m) + \rho_f\nabla_{\mathbf{x}}h_f\big)\\
&\quad -(\rho_s-\rho_f)g\cos\theta\,\frac{h_m}{2}\nabla_{\mathbf{x}}\bar\varphi + (1-\bar\varphi)\overline{\nabla_{\mathbf{x}}p^e_{f_m}}\\
&\quad -\mathrm{sgn}\,(\overline{v^{\mathbf{x}}})\tan\overline{\delta_{\mathrm{eff}}}\,\frac{\big(\bar\varphi(\rho_s-\rho_f)g\cos\theta h_m - (p^e_{f_m})_{|b}\big)_+}{h_m}\\
&\quad +\bar\beta(\overline{u^{\mathbf{x}}}-\overline{v^{\mathbf{x}}}) - \bar\varphi\rho_s g\sin\theta(1,0)^t,
\end{aligned}
$$

$$
\begin{aligned}
\partial_t\big((1-\bar\varphi)h_m\big) + \nabla_{\mathbf{x}}\cdot\big((1-\bar\varphi)h_m\overline{u^{\mathbf{x}}}\big) &= -\mathcal{V}_f,\\
\rho_f(1-\bar\varphi)\big(\partial_t\overline{u^{\mathbf{x}}} + \overline{u^{\mathbf{x}}}\cdot\nabla_{\mathbf{x}}\overline{u^{\mathbf{x}}}\big) &= -(1-\bar\varphi)\rho_f g\cos\theta\nabla_{\mathbf{x}}(b+h_m+h_f)\\
&\quad -(1-\bar\varphi)\overline{\nabla_{\mathbf{x}}p^e_{f_m}}\\
&\quad -\frac{1}{h_m}\Big(\big(\frac{1}{2}\rho_f\mathcal{V}_f - k_i\big)(\overline{u^{\mathbf{x}}_f}-\overline{u^{\mathbf{x}}}) + k_b\overline{u^{\mathbf{x}}}\Big)\\
&\quad -\bar\beta(\overline{u^{\mathbf{x}}}-\overline{v^{\mathbf{x}}}) - (1-\bar\varphi)\rho_f g\sin\theta(1,0)^t,
\end{aligned}
$$

The two-phase tow-layer model

Where:

$$\overline{\nabla_{\mathbf{x}} p^e_{fm}} = \frac{1}{h_m}\Big(\nabla_{\mathbf{x}}(h_m \overline{p^e_{fm}}) + (p^e_{fm})_{|b}\nabla_{\mathbf{x}} b\Big),$$

and

- Case (I) $(\bar{\beta} = O(\epsilon^{-1}))$:

$$(p^e_{fm})_{|b} = -\frac{\bar{\beta}}{(1-\bar{\varphi})^2}\frac{h_m^2}{2}\bar{\Phi}, \qquad \overline{p^e_{fm}} = -\frac{\bar{\beta}}{(1-\bar{\varphi})^2}\frac{h_m^2}{3}\bar{\Phi}$$

- Case (II) $(\bar{\beta} = O(1))$:

$$(p^e_{fm})_{|b} = -\frac{\bar{\beta}}{1-\bar{\varphi}}\left(\frac{h_m^2}{2}\frac{\bar{\Phi} + \nabla_{\mathbf{x}}\cdot\big((1-\bar{\varphi})(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}})\big)}{1-\bar{\varphi}} - h_m(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}})\cdot\nabla_{\mathbf{x}} b\right),$$

$$\overline{p^e_{fm}} = -\frac{\bar{\beta}}{1-\bar{\varphi}}\left(\frac{h_m^2}{3}\frac{\bar{\Phi} + \nabla_{\mathbf{x}}\cdot\big((1-\bar{\varphi})(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}})\big)}{1-\bar{\varphi}} - \frac{h_m}{2}(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}})\cdot\nabla_{\mathbf{x}} b\right).$$

Parameter settings

- Friction coefficient:

$$\bar{\beta} = (1-\bar{\varphi})^2 \frac{\eta_f}{\bar{\kappa}}, \qquad \bar{\kappa} = \frac{d^2(1-\bar{\varphi})^3}{150\bar{\varphi}^2}.$$

- Effective bottom solid friction:

$$\tan \overline{\delta_{\text{eff}}} = \tan\delta + K(\bar{\varphi} - \bar{\varphi}_c^{eq}).$$

- Dilatance closure:

$$\bar{\Phi} = K\bar{\dot{\gamma}}(\bar{\varphi} - \bar{\varphi}_c^{eq}).$$

- Critical-state compacity $\bar{\varphi}_c^{eq}$:

$$\bar{\varphi}_c^{eq} = \bar{\varphi}_c^{stat} - K_2 \frac{\eta_f \bar{\dot{\gamma}}}{p_{s|b}},$$

- Solid pressure:

$$p_{s|b} = \bar{\varphi}(\rho_s - \rho_f) g \cos\theta h_m - (p_{fm}^e)_{|b}, \quad (p_{fm}^e)_{|b} = -\frac{\bar{\beta}}{(1-\bar{\varphi})^2} \frac{h_m^2}{2} \bar{\Phi},$$

The energy balance associated to Jackson's system

$$
\begin{aligned}
&\partial t \left(\rho_s \varphi \frac{|v|^2}{2} + \rho_f (1-\varphi) \frac{|u|^2}{2} - (\mathbf{g} \cdot X)\big(\rho_s \varphi + \rho_f (1-\varphi)\big) \right) \\
&+\nabla \cdot \Big(\rho_s \varphi \frac{|v|^2}{2} v + \rho_f (1-\varphi) \frac{|u|^2}{2} u - (\mathbf{g} \cdot X)\big(\rho_s \varphi v + \rho_f (1-\varphi) u\big) \\
&\qquad + p_{f_m} \big(\varphi v + (1-\varphi) u\big) + \widetilde{T_{f_m}} u + T_s v \Big) \\
&= T_s : \nabla v + \widetilde{T_{f_m}} : \nabla u + f \cdot (v-u),
\end{aligned}
$$

where X denotes the space position.

- The friction effects give naturally a dissipative term $f \cdot (v-u) \leq 0$,
- it is also natural to assume that $\widetilde{T_{f_m}} : \nabla u \leq 0$.
- Moreover:

$$T_s : \nabla v = p_s \nabla \cdot v + \widetilde{T_s} : \nabla v,$$

  - It is also natural to have $\widetilde{T_s} : \nabla v \leq 0$,
  - it remains the term $p_s \nabla \cdot v$.

Closure:

$$\nabla \cdot v = \Phi$$

The energy balance associated to Jackson's system

The right hand side of the 3D energy balance is written as

$$R_s = p_s\Phi + \widetilde{T}_s : \nabla v + \widetilde{T}_{f_m} : \nabla u + f \cdot (v - u)$$

where

$$\Phi = K\dot{\gamma}(\varphi - \varphi_c^{eq}).$$

The energy balance associated to Jackson's system

$$R_s = p_s K \dot{\gamma}(\varphi - \varphi_c^{eq}) + \widetilde{T}_s : \nabla v + \widetilde{T}_{f_m} : \nabla u + f \cdot (v - u)$$

- If $\varphi < \varphi_c$ then the granular medium contracts ($\nabla \cdot v < 0$) as soon as there is a deformation ($\dot{\gamma} > 0$). Consequently,
  - water must be expelled from the mixture,
  - the pore pressure increases.
  - Friction decreases.

  then $p_s \Phi = p_s K \dot{\gamma}(\varphi - \varphi_c^{eq}) \leq 0$ (at least if p_s remains positive), and R_s is clearly nonpositive.

The energy balance associated to Jackson's system

$$R_s = p_s K \dot{\gamma}(\varphi - \varphi_c^{eq}) + \widetilde{T}_s : \nabla v + \widetilde{T}_{fm} : \nabla u + f \cdot (v - u)$$

- If $\varphi > \varphi_c^{eq}$ then the granular medium dilates ($\nabla \cdot v > 0$) as soon as there is a deformation ($\dot{\gamma} > 0$). Consequently,
  - water must be sucked by the mixture,
  - the pore pressure decreases.
  - Friction increases.

  then $p_s \Phi \geq 0$. Thus, in this case, the friction forces need to be strong enough to balance the energy of the system. Namely, the internal friction between solid particles must generate a dissipation $\widetilde{T}_s : \nabla v$ sufficiently negative such that, together with the friction in the mixture $f \cdot (v - u)$, counterbalance the previous term $p_s \Phi$.

The energy balance associated to Jackson's system

$$R_s = p_s K \dot{\gamma}(\varphi - \varphi_c^{eq}) + \widetilde{T}_s : \nabla v + \widetilde{T}_{f_m} : \nabla u + f \cdot (v - u)$$

- Interpretation as a compressible model
  We propose an interpretation of the dilatancy relation as a compressible model, that enables to write down a fully dissipative energy equation in the case when the critical-state compacity φ_c^{eq} depends only on the pressure p_s, and not on $\dot{\gamma}$.

The energy balance associated to Jackson's system

$$R_s = p_s K \dot{\gamma}(\varphi - \varphi_c^{eq}) + \widetilde{T}_s : \nabla v + \widetilde{T}_{f_m} : \nabla u + f \cdot (v - u)$$

- Interpretation as a compressible model
  We consider the critical volume fraction φ_c^{eq} to be an increasing function of the solid pressure only, $\varphi_c^{eq} = \varphi_c^{eq}(p_s)$, bounded by some maximal value φ_{max}.

  This function $\varphi = \varphi_c^{eq}(p_s)$ can be defined by its inverse $p = p_c^{eq}(\varphi)$ ($p_c^{eq}(\varphi)$ being called the critical pressure).

Interpretation as a compressible model

$$\varphi_c^{eq} = \varphi_c^{eq}(p_s), \qquad (p = p_c^{eq}(\varphi))$$

The energy equation gives

$$\begin{aligned}
&\partial t \left(\rho_s \varphi \frac{|v|^2}{2} + \rho_f (1-\varphi) \frac{|u|^2}{2} - (\mathbf{g} \cdot X)\big(\rho_s \varphi + \rho_f (1-\varphi)\big) + \rho_s \varphi e_c^{eq} \right) \\
&+\nabla \cdot \Bigg(\rho_s \varphi \frac{|v|^2}{2} v + \rho_f (1-\varphi) \frac{|u|^2}{2} u - (\mathbf{g} \cdot X)\big(\rho_s \varphi v + \rho_f (1-\varphi) u\big) \\
&\qquad + p_{fm} \big(\varphi v + (1-\varphi) u\big) + \widetilde{T_{fm}} u + T_s v + \rho_s \varphi e_c^{eq} v \Bigg) \\
&= (p_s - p_c^{eq}) K \dot{\gamma} (\varphi - \varphi_c^{eq}) + \widetilde{T}_s : \nabla v + \widetilde{T_{fm}} : \nabla u + f \cdot (v - u).
\end{aligned}$$

Since $p_s - p_c^{eq}(\varphi)$ and $\varphi - \varphi_c^{eq}(p_s)$ have opposite signs – because φ_c^{eq} is an increasing function of p_s– one has $(p_s - p_c^{eq}) \nabla \cdot v \leq 0$, and the energy balance equation (31) has a nonpositive right-hand side.

Interpretation as a compressible model

$$\varphi_c^{eq} = \varphi_c^{eq}(p_s), \qquad (p = p_c^{eq}(\varphi))$$

The energy equation gives

$$\begin{aligned}
&\partial t \left(\rho_s \varphi \frac{|v|^2}{2} + \rho_f (1-\varphi) \frac{|u|^2}{2} - (\mathbf{g} \cdot X)\big(\rho_s \varphi + \rho_f (1-\varphi)\big) + \rho_s \varphi e_c^{eq} \right) \\
&+\nabla \cdot \left(\rho_s \varphi \frac{|v|^2}{2} v + \rho_f (1-\varphi) \frac{|u|^2}{2} u - (\mathbf{g} \cdot X)\big(\rho_s \varphi v + \rho_f (1-\varphi) u\big) \right. \\
&\qquad \left. + p_{f_m} \big(\varphi v + (1-\varphi) u\big) + \widetilde{T_{f_m}} u + T_s v + \rho_s \varphi e_c^{eq} v \right) \\
&= (p_s - p_c^{eq}) K \dot{\gamma} (\varphi - \varphi_c^{eq}) + \widetilde{T}_s : \nabla v + \widetilde{T_{f_m}} : \nabla u + f \cdot (v - u).
\end{aligned}$$

Remark: Classically in thermodynamics, the mechanical internal energy U is related to the pressure p and volume V by the relation $\mathrm{d}U = -p\mathrm{d}V$. Here the specific volume (i.e. volume per mass unit) is $1/(\rho_s \varphi)$, thus to the critical pressure $p_c^{eq}(\varphi)$ one can associate by this relation a specific internal energy (i.e. internal energy per mass unit) $e_c^{eq}(\varphi)$. Since $\mathrm{d}(1/\varphi) = -\mathrm{d}\varphi/\varphi^2$ we obtain the differential relation

$$\frac{\mathrm{d}e_c^{eq}}{\mathrm{d}\varphi} = \frac{p_c^{eq}}{\rho_s \varphi^2}.$$

Energy balance for the proposed model

Then one has the following local energy balance identity,

$$
\begin{aligned}
\partial t \Bigg(& \rho_s \bar{\varphi} h_m \frac{|\overline{v^{\mathbf{x}}}|^2}{2} + \rho_f (1-\bar{\varphi}) h_m \frac{|\overline{u^{\mathbf{x}}}|^2}{2} + \rho_f h_f \frac{|\overline{u_f^{\mathbf{x}}}|^2}{2} + \rho_s h_m \bar{\varphi} e_c^{eq}(\bar{\varphi}) \\
& + g \cos\theta \Big(\rho_s \bar{\varphi} h_m + \rho_f \big((1-\bar{\varphi}) h_m + h_f \big) \Big) (b + \tilde{b}) \\
& + (\rho_s - \rho_f) g \cos\theta \bar{\varphi} \frac{h_m^2}{2} + \rho_f g \cos\theta \frac{(h_m + h_f)^2}{2} \Bigg) \\
+ \nabla_{\mathbf{x}} \cdot \Bigg(& \rho_s \bar{\varphi} h_m \frac{|\overline{v^{\mathbf{x}}}|^2}{2} \overline{v^{\mathbf{x}}} + \rho_f (1-\bar{\varphi}) h_m \frac{|\overline{u^{\mathbf{x}}}|^2}{2} \overline{u^{\mathbf{x}}} + \rho_f h_f \frac{|\overline{u_f^{\mathbf{x}}}|^2}{2} \overline{u_f^{\mathbf{x}}} + \rho_s h_m \bar{\varphi} e_c^{eq}(\bar{\varphi}) \overline{v^{\mathbf{x}}} \\
& + g \cos\theta \Big(\rho_s \bar{\varphi} h_m \overline{v^{\mathbf{x}}} + \rho_f \big((1-\bar{\varphi}) h_m \overline{u^{\mathbf{x}}} + h_f \overline{u_f^{\mathbf{x}}} \big) \Big) (b + \tilde{b} + h_m) \\
& + \rho_f g \cos\theta \Big(\bar{\varphi} h_m \overline{v^{\mathbf{x}}} + (1-\bar{\varphi}) h_m \overline{u^{\mathbf{x}}} + h_f \overline{u_f^{\mathbf{x}}} \Big) h_f + (1-\bar{\varphi}) h_m \overline{p_{fm}^{e}} (\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \Bigg) \\
= R, &
\end{aligned}
$$

Energy balance for the proposed model

$$R = (p_s - p_c^{eq})\underbrace{K\dot{\gamma}(\varphi - \varphi_c^{eq})}_{\Phi} + \widetilde{T}_s : \nabla v + \widetilde{T_{fm}} : \nabla u + f \cdot (v - u).$$

Where

$$R = \left(\overline{p}_s - p_c^{eq}(\bar{\varphi})\right)h_m\bar{\Phi} + h_m\overline{p_{fm}^e}\,\bar{\Phi} + R_e - \bar{\beta}h_m|\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}|^2$$
$$- |\overline{v^{\mathbf{x}}}|\tan\overline{\delta_{\text{eff}}}\left(\bar{\varphi}(\rho_s - \rho_f)g\cos\theta h_m - (p_{fm}^e)_{|b}\right)_+ - k_i|\overline{u_f^{\mathbf{x}}} - \overline{u^{\mathbf{x}}}|^2 - k_b|\overline{u^{\mathbf{x}}}|^2,$$

Energy balance for the proposed model

$$R = (p_s - p_c^{eq})\underbrace{K\dot{\gamma}(\varphi - \varphi_c^{eq})}_{\Phi} + \widetilde{T}_s : \nabla v + \widetilde{T_{fm}} : \nabla u + f \cdot (v - u).$$

Where

$$R = \left(\overline{p}_s - p_c^{eq}(\bar{\varphi})\right) h_m \bar{\Phi} + h_m \overline{p_{fm}^e}\,\bar{\Phi} + R_e - \bar{\beta} h_m |\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}|^2$$
$$- |\overline{v^{\mathbf{x}}}| \tan \overline{\delta_{\text{eff}}} \left(\bar{\varphi}(\rho_s - \rho_f) g \cos\theta h_m - (p_{fm}^e)_{|b}\right)_+ - k_i |\overline{u_f^{\mathbf{x}}} - \overline{u^{\mathbf{x}}}|^2 - k_b |\overline{u^{\mathbf{x}}}|^2,$$

with

$$R_e = h_m \overline{p_{fm}^e} \nabla_{\mathbf{x}} \cdot \Big((1 - \bar{\varphi})(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}})\Big) - (1 - \bar{\varphi})(p_{fm}^e)_{|b}(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \cdot \nabla_{\mathbf{x}} b,$$

Energy balance for the proposed model

$$R = (p_s - p_c^{eq}) \underbrace{K\dot{\gamma}(\varphi - \varphi_c^{eq})}_{\Phi} + \widetilde{T}_s : \nabla v + \widetilde{T_{fm}} : \nabla u + f \cdot (v - u).$$

Where

$$R = \left(\bar{p}_s - p_c^{eq}(\bar{\varphi})\right) h_m \bar{\Phi} + h_m \overline{p_{fm}^e}\,\bar{\Phi} + R_e - \bar{\beta} h_m |\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}|^2$$
$$- |\overline{v^{\mathbf{x}}}| \tan \overline{\delta_{\text{eff}}} \left(\bar{\varphi}(\rho_s - \rho_f) g \cos\theta h_m - (p_{fm}^e)_{|b}\right)_+ - k_i |\overline{u_f^{\mathbf{x}}} - \overline{u^{\mathbf{x}}}|^2 - k_b |\overline{u^{\mathbf{x}}}|^2,$$

But:

$$h_m \overline{p_{fm}^e}\,\bar{\Phi} + R_e = -\bar{\beta} \int_b^{b+h_m} (u^z - v^z)^2 dz \quad \text{in case (II)},$$

while further error in $O(\epsilon^3)$ need to be added in case (I).

The immersed configuration

To simulate underwater granular flows, we take the upper pure fluid layer at rest $\overline{u_f^{\mathbf{x}}} = 0$ in our three-velocity model and

$$h_m(t) + h_f(t,x) + x\tan\theta = cst,$$

The equations are then :

$$\partial_t(\bar{\varphi} h_m) + ... = 0, \quad \partial_t \bar{\varphi} + ... = -\bar{\varphi}\bar{\Phi}, \tag{10}$$

$$\rho_s \bar{\varphi} \partial_t \overline{v^{\mathbf{x}}} + ... = -\mathrm{sgn}\left(\overline{v^{\mathbf{x}}}\right) \frac{\tau_b}{h_m} + \bar{\beta}(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) - \bar{\varphi}(\rho_s - \rho_f) g \sin\theta, \tag{11}$$

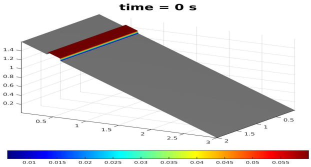
$$\rho_f (1-\bar{\varphi}) \partial_t \overline{u^{\mathbf{x}}} + ... = \left(\frac{1}{2}\rho_f \mathcal{V}_f - k_b\right) \frac{\overline{u^{\mathbf{x}}}}{h_m} - \bar{\beta}(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}), \tag{12}$$

with

$$\mathcal{V}_f = -h_m \bar{\Phi}, \quad \tau_b = \tan \overline{\delta_{\mathrm{eff}}}\, p_{s|b} + K_1 \eta_f \bar{\dot{\gamma}}, \tag{13}$$

$$p_{s|b} = \bar{\varphi}(\rho_s - \rho_f) g \cos\theta h_m - (p^e_{fm})_{|b}, \quad (p^e_{fm})_{|b} = -\frac{\bar{\beta}}{(1-\bar{\varphi})^2} \frac{h_m^2}{2} \bar{\Phi}, \tag{14}$$

Inmersed non-uniform test



(a) Initial condition

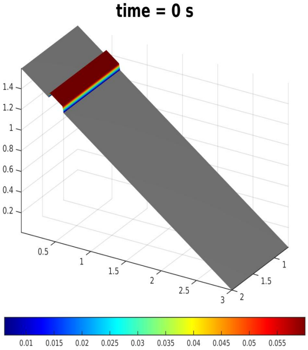
The initial conditions are

$$\overline{u^{\mathbf{x}}}(t=0) = 0\,\text{m/s}, \quad \overline{v^{\mathbf{x}}}(t=0) = 0\,\text{m/s}, \quad \bar{\varphi}(t=0) = \bar{\varphi}^0, \quad h_m(t=0) = h_m^0. \tag{15}$$

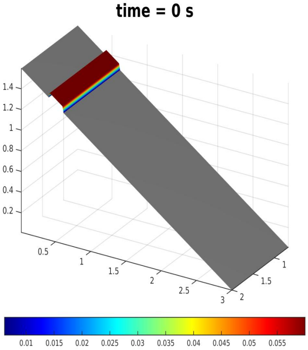
$$\begin{aligned} \rho_s = 2500\,\text{kg/m}^3, \quad \bar{\varphi}_c^{stat} = 0.582, \quad \tan\delta = 0.415, \quad d = 160\mu\text{m}, \\ K = 4.09, \quad K_1 = 90.5, \quad K_2 = 25, \end{aligned} \tag{16}$$

- Low viscosity: $\eta_f = 9.8 \times 10^{-3}\,\text{Pa}\cdot\text{s}$, $\rho_f = 1026\,\text{kg/m}^3$, $|\theta| = 28^{\text{o}}$, $h_m^0 = 6.1$ mm. For the dense case: $\bar{\varphi}^0 = 0.592$.
- Periodic boundary conditions.

Inmersed non-uniform test

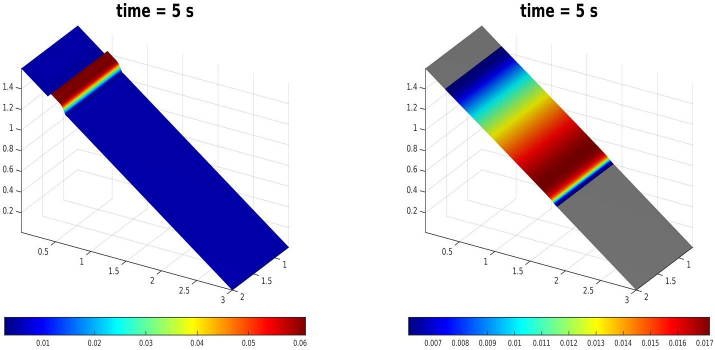


(b) P^{e}_{fm}



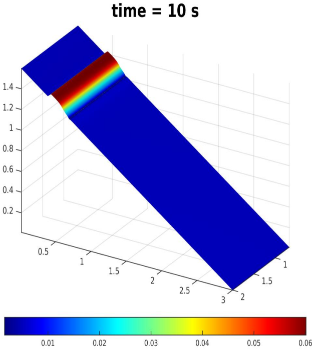
(c) $p^{e}_{fm} = 0$

Inmersed non-uniform test

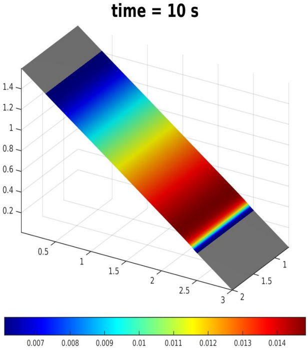


(d) P^e_{fm} (e) $p^e_{fm} = 0$

Inmersed non-uniform test

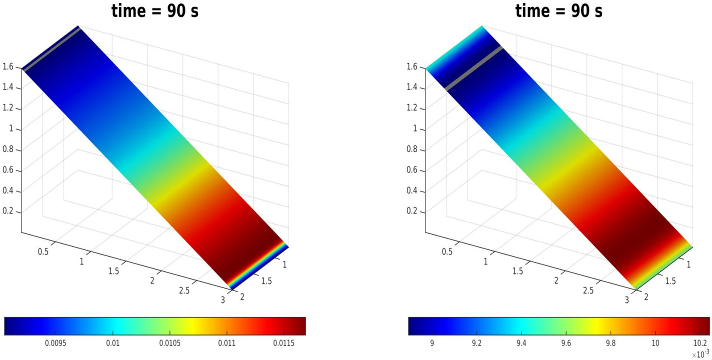


(f) P^e_{fm}



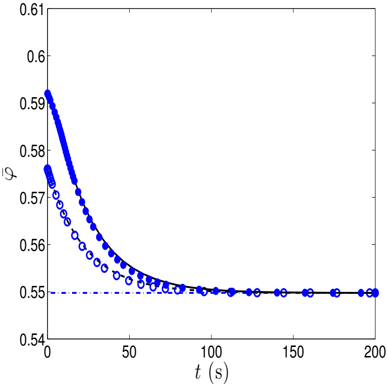
(g) $p^e_{fm} = 0$

Inmersed non-uniform test

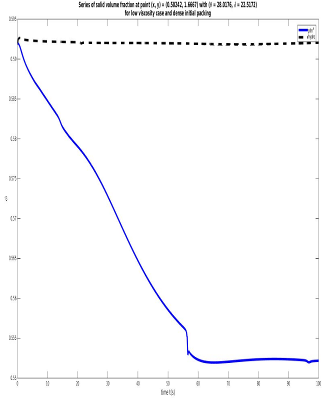


(h) P^e_{fm} (i) $p^e_{fm} = 0$

Inmersed non-uniform test



(j) Uniform flow



(k) Non-uniform $x = 0.5$

Conclusions

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- A two-phase model for debris flows with dilatancy effects with two interfaces has been proposed.
- It is also possible to be used in inmersed configuration.
- Numerical discretization based in IFCP method and a combination of two hydrostatic reconstructions.

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A two-phase solid/fluid model for dense granular flows including dilatancy effects. Part II

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