Formulations multi-couches des équations de Navier-Stokes
Traitement des termes de rhéologie

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avec

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Equations de Navier-Stokes hydrostatique incompressible en 2D

- $\mathbf{u} = (u, w)^T$ vecteur vitesse, p pression du fluide

Conservation de la masse

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Conservation de la quantité de mouvement (g : gravité)

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial \Sigma_{xx}}{\partial x} + \frac{\partial \Sigma_{xz}}{\partial z}$$

$$\frac{\partial p}{\partial z} = -g + \frac{\partial \Sigma_{zx}}{\partial x} + \frac{\partial \Sigma_{zz}}{\partial z}$$

- Tenseur des contraintes visqueuses : $\Sigma_T = -pI_d + \Sigma$

$$\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xz} \\ \Sigma_{zx} & \Sigma_{zz} \end{pmatrix} = \text{rhéologie du fluide}$$

- Domaine espace-temps

$$t > t_0, \quad x \in \mathbb{R}, \quad z_b(x) \leq z \leq \eta(x, t),$$

η cote de la surface libre, z_b le fond, $H := \eta - z_b$ hauteur de fluide

Conditions aux limites cinématiques et dynamiques

Surface libre

$$\frac{\partial \eta}{\partial t} + u_s \frac{\partial \eta}{\partial x} - w_s = 0$$

$$\Sigma_T \mathbf{n}_s = -p^a \mathbf{n}_s \;\Leftrightarrow\; \left((p^a - p) I_d + \Sigma \right) \mathbf{n}_s = 0$$

on prendra $p^a(x,t) = p^a = 0$

Fond

$$\mathbf{u}_b \cdot \mathbf{n}_b = 0 \quad \text{or} \quad u_b \frac{\partial z_b}{\partial x} - w_b = 0$$

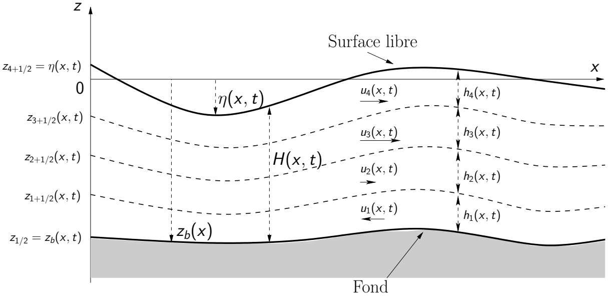
$$\mathbf{t}_b \cdot \Sigma \mathbf{n}_b = \frac{\kappa}{c_b} u_b$$

avec $\kappa = \kappa(\mathbf{u_b}, H) \geq 0$ et $c_b = \dfrac{1}{\sqrt{1 + \left(\frac{\partial z_b}{\partial x}\right)^2}}$

Discrétisation verticale multicouches du domaine fluide

$[z_b, \eta] = N$ couches $(L_\alpha)_{\alpha\in\{1,\dots,N\}}$ d'épaisseur $l_\alpha H(x,t)$; $l_\alpha > 0$, $\sum_{\alpha=1}^N l_\alpha = 1$

$$L_\alpha(x,t) = \{\ x \in \mathbb{R},\quad t > t_0,\quad z \in [z_{\alpha-1/2}, z_{\alpha+1/2}]\ \}$$



$$\begin{cases} z_{\alpha+1/2}(x,t) = z_b(x) + \sum_{j=1}^{\alpha} l_j H(x,t) \\ z_\alpha(x,t) = \dfrac{z_{\alpha+1/2}(x,t) + z_{\alpha-1/2}(x,t)}{2} \\ h_\alpha(x,t) = z_{\alpha+1/2}(x,t) - z_{\alpha-1/2}(x,t) = l_\alpha H(x,t) \\ h_{\alpha+1/2}(x,t) = z_{\alpha+1}(x,t) - z_\alpha(x,t) = \dfrac{h_{\alpha+1}(x,t) + h_\alpha(x,t)}{2} \end{cases}$$

Prise de moyenne verticale de Navier-Stokes hydrostatique incompressible...

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + g\frac{\partial \eta}{\partial x} = \frac{\partial \Sigma_{xx}}{\partial x} + \frac{\partial \Sigma_{xz}}{\partial z} + \frac{\partial^2}{\partial x^2}\int_z^\eta \Sigma_{zx}\, dz_1 - \frac{\partial \Sigma_{zz}}{\partial x}$$

$$p = g(\eta - z) - \frac{\partial}{\partial x}\int_z^\eta \Sigma_{zx}\, dz_1 + \Sigma_{zz}$$

... $\Rightarrow$ formulation multicouches d'inconnues H et $(u_\alpha)_{\alpha=1,\ldots,N}$

$$\sum_{\alpha=1}^{N} \frac{\partial h_\alpha}{\partial t} + \sum_{\alpha=1}^{N} \frac{\partial (h_\alpha u_\alpha)}{\partial x} = 0$$

$$\frac{\partial h_\alpha u_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(h_\alpha u_\alpha^2 + \frac{g}{2} h_\alpha H\right) = -g h_\alpha \frac{\partial z_b}{\partial x} + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2}$$
$$+ \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left(\frac{\partial \Sigma_{xx}}{\partial x} + \frac{\partial \Sigma_{xz}}{\partial z} + \frac{\partial^2}{\partial x^2}\int_z^\eta \Sigma_{zx}\, dz_1 - \frac{\partial \Sigma_{zz}}{\partial x}\right) dz$$

$$w_\alpha = -\frac{1}{2}\frac{\partial (h_\alpha u_\alpha)}{\partial x} - \sum_{j=1}^{\alpha-1} \frac{\partial (h_j u_j)}{\partial x} + u_\alpha \frac{\partial z_\alpha}{\partial x}$$

Définitions, relations de quadrature et de fermeture

$$\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \Sigma_{ab}\ dz \approx h_\alpha \Sigma_{ab,\alpha}$$

$$\Sigma_{ab|_{z_{\alpha+1/2}}} \approx \Sigma_{ab,\alpha+1/2}$$

$$\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} z\Sigma_{zx}\ dz \approx h_\alpha z_\alpha \Sigma_{zx,\alpha}$$

⋆ Localisation des Σ_{ab} au sein des couches α ou aux interfaces $\alpha + 1/2$? ⋆

- Condition à la surface libre: $\mathbf{t}_{N+1/2} \cdot \Sigma_{N+1/2} \mathbf{n}_{N+1/2} = 0$
- Condition au fond: $\mathbf{t}_{1/2} \cdot \Sigma_{1/2} \mathbf{n}_{1/2} = \dfrac{\kappa}{c_b} u_1$

Approximation des termes visqueux dans le système multicouches

$$\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left(\frac{\partial \Sigma_{xx}}{\partial x} - \frac{\partial \Sigma_{zz}}{\partial x} + \frac{\partial^2}{\partial x^2} \int_z^{\eta} \Sigma_{zx} dz_1 + \frac{\partial \Sigma_{xz}}{\partial z} \right) dz$$

$$\approx$$

$$\frac{\partial}{\partial x} h_\alpha \left(\Sigma_{xx,\alpha} - \Sigma_{zz,\alpha} \right)$$

$$- \frac{\partial z_{\alpha+1/2}}{\partial x} \left(\Sigma_{xx,\alpha+1/2} - \Sigma_{zz,\alpha+1/2} \right) + \frac{\partial z_{\alpha-1/2}}{\partial x} \left(\Sigma_{xx,\alpha-1/2} - \Sigma_{zz,\alpha-1/2} \right)$$

$$+ \frac{\partial^2}{\partial x^2} \left(h_\alpha z_\alpha \Sigma_{zx,\alpha} \right) + z_{\alpha+1/2} \frac{\partial^2}{\partial x^2} \left(\sum_{j=\alpha+1}^{N} h_j \Sigma_{zx,j} \right) - \frac{\partial^2 z_{\alpha+1/2}}{\partial x^2} \Sigma_{zx,\alpha+1/2}$$

$$- z_{\alpha-1/2} \frac{\partial^2}{\partial x^2} \left(\sum_{j=\alpha}^{N} h_j \Sigma_{zx,j} \right) + \frac{\partial^2 z_{\alpha-1/2}}{\partial x^2} \Sigma_{zx,\alpha-1/2}$$

$$+ \Sigma_{xz,\alpha+1/2} - \Sigma_{xz,\alpha-1/2}$$

Problème continu - $E = \frac{u^2}{2} + gz$

$$\frac{\partial}{\partial t}\int_{z_b}^{\eta} E\; dz + \frac{\partial}{\partial x}\int_{z_b}^{\eta}\left[u\left(E + g(\eta - z) - (\Sigma_{xx} - \Sigma_{zz}) - \frac{\partial}{\partial x}\int_{z}^{\eta}\Sigma_{zx} dz_1\right) - w\Sigma_{zx}\right] dz$$

$$= -\int_{z_b}^{\eta}\left(\frac{\partial u}{\partial x}(\Sigma_{xx} - \Sigma_{zz}) + \frac{\partial w}{\partial x}\Sigma_{zx} + \frac{\partial u}{\partial z}\Sigma_{xz}\right) dz - \frac{\kappa}{c_b^3} u_b^2$$

Système multicouches - $E_\alpha^N = \frac{u_\alpha^2}{2} + gz_\alpha$

$$\frac{\partial}{\partial t}\left(\sum_{\alpha=1}^{N} h_\alpha E_\alpha^N\right) + \frac{\partial}{\partial x}\left(\sum_{\alpha=1}^{N} h_\alpha \left[u_\alpha\left(E_\alpha^N + \frac{g}{2}H - (\Sigma_{xx,\alpha} - \Sigma_{zz,\alpha})\right.\right.\right.$$

$$\left.\left.\left. - \left(\frac{\partial z_\alpha}{\partial x}\Sigma_{zx,\alpha} + \frac{\partial}{\partial x}\left(\frac{h_\alpha}{2}\Sigma_{zx,\alpha} + \sum_{j=\alpha+1}^{N} h_j \Sigma_{zx,j}\right)\right)\right) - w_\alpha \Sigma_{zx,\alpha}\right]\right)$$

$$= -\sum_{\alpha=1}^{N}\left(\frac{\partial u_\alpha}{\partial x} h_\alpha\left(\Sigma_{xx,\alpha} - \Sigma_{zz,\alpha}\right) - \left(\frac{\partial z_{\alpha+1/2}}{\partial x}\left(\Sigma_{xx,\alpha+1/2} - \Sigma_{zz,\alpha+1/2}\right)\right)(u_{\alpha+1} - u_\alpha)\right.$$

$$+ h_\alpha \Sigma_{zx,\alpha}\left(\frac{\partial w_\alpha}{\partial x} + \frac{\partial z_\alpha}{\partial x}\frac{\partial u_\alpha}{\partial x}\right) - \left(\frac{\partial z_{\alpha+1/2}}{\partial x}\right)^2 \Sigma_{zx,\alpha+1/2}(u_{\alpha+1} - u_\alpha)$$

$$\left. + \Sigma_{xz,\alpha+1/2}(u_{\alpha+1} - u_\alpha)\right) - \frac{\kappa}{c_b^3} u_1^2$$

Consistance des termes visqueux dans l'équation d'énergie (1/5)

LHS: termes en Σ_{xz}, Σ_{zz} et Σ_{zx}

$$\int_{z_b}^{\eta} \left(-u(\Sigma_{xx} - \Sigma_{zz}) - w\Sigma_{zx} \right) dz \quad \simeq \quad \sum_{\alpha=1}^{N} h_\alpha \left(-u_\alpha(\Sigma_{xx,\alpha} - \Sigma_{zz,\alpha}) - w_\alpha \Sigma_{zx,\alpha} \right)$$

LHS : termes en Σ_{zx}

$$\begin{aligned}
\int_{z_b}^{\eta} u \frac{\partial}{\partial x} \int_{z}^{\eta} \Sigma_{zx} dz_1 dz &= \sum_{\alpha=1}^{N} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u \frac{\partial}{\partial x} \int_{z}^{\eta} \Sigma_{zx} dz_1 dz \\
&\simeq \sum_{\alpha=1}^{N} h_\alpha u_\alpha \left(\frac{\partial}{\partial x} \int_{z}^{\eta} \Sigma_{zx} dz_1 \right)_{|z=z_\alpha}
\end{aligned}$$

avec

$$\begin{aligned}
\left(\frac{\partial}{\partial x} \int_{z}^{\eta} \Sigma_{zx} dz_1 \right)_{|z=z_\alpha} &= \frac{\partial}{\partial x} \int_{z_\alpha}^{\eta} \Sigma_{zx} dz_1 + \frac{\partial z_\alpha}{\partial x} \Sigma_{zx,z=z_\alpha} \\
&\simeq \frac{\partial}{\partial x} \left(\frac{h_\alpha}{2} \Sigma_{zx,\alpha} + \sum_{j=\alpha+1}^{N} h_j \Sigma_{zx,j} \right) + \frac{\partial z_\alpha}{\partial x} \Sigma_{zx,\alpha}
\end{aligned}$$

RHS: termes en Σ_{xz}

$$\int_{z_b}^{\eta} \frac{\partial u}{\partial z} \Sigma_{xz} dz \simeq \sum_{\alpha=1}^{N} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\partial u}{\partial z} \Sigma_{xz} dz$$

$$\simeq \sum_{\alpha=1}^{N} \frac{h_\alpha}{2} \left(\Sigma_{xz,\alpha+1/2} \left(\frac{\partial u}{\partial z}\right)_{|z_{\alpha+1/2}} + \Sigma_{xz,\alpha-1/2} \left(\frac{\partial u}{\partial z}\right)_{|z_{\alpha-1/2}} \right)$$

$$\simeq \sum_{\alpha=1}^{N} \frac{h_\alpha}{2} \left(\Sigma_{xz,\alpha+1/2} \frac{u_{\alpha+1} - u_\alpha}{(h_\alpha + h_{\alpha+1})/2} + \Sigma_{xz,\alpha-1/2} \frac{u_\alpha - u_{\alpha-1}}{(h_{\alpha-1} + h_\alpha)/2} \right)$$

RHS: termes en Σ_{xx} et Σ_{zz}

$$\frac{\partial u}{\partial x}_{|z=z_\alpha} = \frac{\partial u_{|z=z_\alpha}}{\partial x} - \frac{\partial z_\alpha}{\partial x}\frac{\partial u}{\partial z}_{|z=z_\alpha}$$

donc

$$\int_{z_b}^{\eta} \frac{\partial u}{\partial x}(\Sigma_{xx} - \Sigma_{zz})dz \simeq \sum_{\alpha=1}^{N}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\partial u}{\partial x}_{|z=z_\alpha}(\Sigma_{xx} - \Sigma_{zz})dz$$

$$\simeq \sum_{\alpha=1}^{N}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left(\frac{\partial u_{|z=z_\alpha}}{\partial x} - \frac{\partial z_\alpha}{\partial x}\frac{\partial u}{\partial z}_{|z=z_\alpha}\right)(\Sigma_{xx} - \Sigma_{zz})dz$$

$$\simeq \sum_{\alpha=1}^{N} h_\alpha\left(\frac{\partial u_\alpha}{\partial x}\left(\Sigma_{xx,\alpha} - \Sigma_{zz,\alpha}\right) - \frac{1}{2}\left(\frac{\partial z_{\alpha+1/2}}{\partial x}\left(\Sigma_{xx,\alpha+1/2} - \Sigma_{zz,\alpha+1/2}\right)\frac{u_{\alpha+1} - u_\alpha}{(h_\alpha + h_{\alpha+1})/2} + \frac{\partial z_{\alpha-1/2}}{\partial x}\left(\Sigma_{xx,\alpha-1/2} - \Sigma_{zz,\alpha-1/2}\right)\frac{u_\alpha - u_{\alpha-1}}{(h_{\alpha-1} + h_\alpha)/2}\right)\right)$$

Membre de droite, termes en Σ_{zx}

$$\frac{\partial w}{\partial x}_{|z=z_\alpha} = \frac{\partial w_{|z=z_\alpha}}{\partial x} - \frac{\partial z_\alpha}{\partial x}\frac{\partial w}{\partial z}_{|z=z_\alpha}$$

$$= \frac{\partial w_{|z=z_\alpha}}{\partial x} + \frac{\partial z_\alpha}{\partial x}\frac{\partial u}{\partial x}_{|z=z_\alpha} = \frac{\partial w_{|z=z_\alpha}}{\partial x} + \frac{\partial z_\alpha}{\partial x}\left(\frac{\partial u_{|z=z_\alpha}}{\partial x} - \frac{\partial z_\alpha}{\partial x}\frac{\partial u}{\partial z}_{|z=z_\alpha}\right)$$

donc

$$\int_{z_b}^{\eta} \frac{\partial w}{\partial x}\Sigma_{zx} dz \simeq \sum_{\alpha=1}^{N}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\partial w}{\partial x}_{|z=z_\alpha}\Sigma_{zx} dz$$

$$\simeq \sum_{\alpha=1}^{N} h_\alpha\Bigg(\left(\frac{\partial w_\alpha}{\partial x} + \frac{\partial z_\alpha}{\partial x}\frac{\partial u_\alpha}{\partial x}\right)\Sigma_{zx,\alpha} - \frac{1}{2}\Bigg(\left(\frac{\partial z_{\alpha+1/2}}{\partial x}\right)^2\Sigma_{zx,\alpha+1/2}\frac{u_{\alpha+1}-u_\alpha}{(h_\alpha+h_{\alpha+1})/2}$$

$$+\left(\frac{\partial z_{\alpha-1/2}}{\partial x}\right)^2\Sigma_{zx,\alpha-1/2}\frac{u_\alpha-u_{\alpha-1}}{(h_{\alpha-1}+h_\alpha)/2}\Bigg)\Bigg)$$

$$\Sigma_{xx} = 2\mu\frac{\partial u}{\partial x} \quad , \quad \Sigma_{zz} = 2\mu\frac{\partial w}{\partial z} \quad , \quad \Sigma_{xz} = \Sigma_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

μ : coefficient de viscosité dynamique

⋆ Quid de la localisation des Σ_{ab} ?

→ Choix 1: au sein des couches α

Variable principale: $\Sigma_{ab,\alpha}$

Variable secondaire: $\Sigma_{ab,\alpha+1/2} = \dfrac{\Sigma_{ab,\alpha+1} + \Sigma_{ab,\alpha}}{2}$

→ Choix 2: aux interfaces $\alpha + 1/2$

Variable principale: $\Sigma_{ab,\alpha+1/2}$

Variable secondaire: $\Sigma_{ab,\alpha} = \dfrac{\Sigma_{ab,\alpha+1/2} + \Sigma_{ab,\alpha-1/2}}{2}$

Choix 1: Cas newtonien avec termes visqueux sur les couches α

- Pour assurer la dissipation de l'énergie :

$$
\begin{aligned}
h_\alpha \Sigma_{xx,\alpha} &= -h_\alpha \Sigma_{zz,\alpha} \\
&= 2\mu \left(h_\alpha \frac{\partial u_\alpha}{\partial x} - \left(\frac{\partial z_{\alpha+1/2}}{\partial x} \frac{u_{\alpha+1} - u_\alpha}{2} + \frac{\partial z_{\alpha-1/2}}{\partial x} \frac{u_\alpha - u_{\alpha-1}}{2} \right) \right)
\end{aligned}
$$

$$
\begin{aligned}
h_\alpha \Sigma_{zx,\alpha} &= h_\alpha \Sigma_{xz,\alpha} \\
&= \mu \left(h_\alpha \frac{\partial w_\alpha}{\partial x} + h_\alpha \frac{\partial z_\alpha}{\partial x} \frac{\partial u_\alpha}{\partial x} + \frac{u_{\alpha+1} - u_\alpha}{2} \left(1 - \left(\frac{\partial z_{\alpha+1/2}}{\partial x} \right)^2 \right) \right. \\
&\qquad \left. + \frac{u_\alpha - u_{\alpha-1}}{2} \left(1 - \left(\frac{\partial z_{\alpha-1/2}}{\partial x} \right)^2 \right) \right)
\end{aligned}
$$

$\Longrightarrow$ RHS du bilan d'énergie :

$$
-\sum_{\alpha=1}^{N} \frac{h_\alpha}{\mu} \left(\Sigma_{xx,\alpha}^2 + \Sigma_{zx,\alpha}^2 \right) - \frac{\kappa}{c_b^3} u_1^2
$$

Choix 2: Cas newtonien avec termes visqueux aux interfaces $\alpha + 1/2$

- Pour assurer la dissipation de l'énergie :

$$\begin{aligned} h_{\alpha+1/2}\Sigma_{xx,\alpha+1/2} &= -h_{\alpha+1/2}\Sigma_{zz,\alpha+1/2} \\ &= 2\mu\ \left(\ \frac{1}{2}\left(h_\alpha \frac{\partial u_\alpha}{\partial x} + h_{\alpha+1}\frac{\partial u_{\alpha+1}}{\partial x}\right) - \frac{\partial z_{\alpha+1/2}}{\partial x}(u_{\alpha+1} - u_\alpha)\ \right) \end{aligned}$$

$$\begin{aligned} h_{\alpha+1/2}\Sigma_{zx,\alpha+1/2} &= h_{\alpha+1/2}\Sigma_{xz,\alpha+1/2} \\ &= \mu\ \left(\ \frac{1}{2}\left(h_\alpha(\frac{\partial w_\alpha}{\partial x} + \frac{\partial z_\alpha}{\partial x}\frac{\partial u_\alpha}{\partial x}) + h_{\alpha+1}(\frac{\partial w_{\alpha+1}}{\partial x} + \frac{\partial z_{\alpha+1}}{\partial x}\frac{\partial u_{\alpha+1}}{\partial x})\right)\right. \\ &\quad \left. +(u_{\alpha+1} - u_\alpha)\left(1 - \left(\frac{\partial z_{\alpha+1/2}}{\partial x}\right)^2\right)\ \right) \end{aligned}$$

$\Longrightarrow$ RHS du bilan d'énergie :

$$-\sum_{\alpha=1}^{N}\left(\frac{h_{\alpha+1/2}}{\mu}(\Sigma^2_{xx,\alpha+1/2} + \Sigma^2_{zx,\alpha+1/2})\right) - \frac{\kappa}{c_b^3}u_1^2$$